§ 5. COUNTABLE AND UNCOUNTABLE SETS

A set S is said to be ***denumerable*** or ***countably infinite*** if there exists a bijection of $\mathbb{N}$ onto S.

A set S is said to be ***countable*** if it is either finite or denumerable.

A set S is said to be ***uncountable*** if it is not countable.

Example. The set $E = \{2n : n \in \mathbb{N}\}$ of even natural numbers is denumerable, since the mapping $f : \mathbb{N} \to E$ defined by $f(n) = 2n$, $n \in \mathbb{N}$ is a bijection of $\mathbb{N}$ onto E. Similarly the set $O = \mathbb{N} \setminus E = \{2n-1 : n \in \mathbb{N}\}$ of odd natural numbers is denumerable.

Remark 1. Two sets A and B are said to be equivalent if there exists a bijection from A onto B. A is equivalent to B is written in symbol as $A \sim B$. Clearly the relation $\sim$ has the following properties :

(i) $A \sim A$ (Reflexive)

(ii) $A \sim B \Rightarrow B \sim A$ (Symmetric)

(iii) $A \sim B, B \sim C \Rightarrow A \sim C$ (Transitive)

Hence the relation $\sim$ is an equivalence relation.

Then the concept of finiteness and countability can also be defined as :

A set S is *finite* if $S \sim \mathbb{N}_n$ for some n, or $S = \varnothing$.

A set S is ***denumerable*** if $S \sim \mathbb{N}$.

Remark 2. A finite set cannot be equivalent to one of its proper subsets. However this is possible for infinite sets as is evident from the example given above in which we have shown that there exists a bijection from $\mathbb{N}$ onto E, the set of even numbers, i.e., $\mathbb{N} \sim E$.

Remark 3. A set A is countable if there exists a 1-1 function f from $\mathbb{N}$ onto A. Then A is the range of f i.e., $A = f(\mathbb{N})$ and hence

$$A = \{f(1), f(2), f(3), \ldots\}.$$

Hence saying that A is countable means that its elements can be "counted" (arranged with labels 1, 2, ...). Instead of $f(1), f(2), f(3), \ldots$ we usually write $a_1, a_2, a_3, \ldots$ which is known as a sequence.

Thus A *is countable if and only if its elements can be arranged as a sequence of distinct items.*

Problem 12. *Show that the set of all integers is denumerable.*

Solution. To construct a bijection from $\mathbb{N}$, the set of natural numb onto the set $\mathbb{Z}$ of all integers, arrange the elements of $\mathbb{N}$ together with elements of $\mathbb{Z}$ as follows :

$$\begin{array}{llllllllll} \mathbb{N}: & 1, & 2, & 3, & 4, & 5, & 6, & 7, & \cdots & \cdots \\ \mathbb{Z}: & 0, & 1, & -1, & 2, & -2, & 3, & -3, & \cdots & \cdots \end{array}$$

From the above arrangement it follows that the function $f: \mathbb{N} \to$ defined by

$$f(n) = \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even} \\ -\dfrac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

is one-to-one and onto. Hence $\mathbb{Z}$, the set of all integers is denumerable

Problem 13. *Show that* $A = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \right\}$ *is denumerable.*

Solution. Define $f : \mathbb{N} \to A$ by

$$f(n) = \frac{n}{n+1}, \quad \text{for all } n \in \mathbb{N}.$$

Under this mapping, $1 \to \frac{1}{2}$, $2 \to \frac{2}{3}$, $3 \to \frac{3}{4}, \ldots$

Thus the mapping f is clearly one-one and onto i.e., f is a bijection fro $\mathbb{N}$ onto A. Hence A is denumerable.

Problem 14. *Show that the set* $\mathbb{N} \times \mathbb{N}$ *is denumerable.*

Solution. Consider the product set $\mathbb{N} \times \mathbb{N}$. The elements of $\mathbb{N} \times$ can be arranged in rows and columns as shown below:

$$\begin{array}{ccccccccc} (1,1) & & (1,2) & & (1,3) & & (1,4) & & \cdots\cdots \\ & \nearrow & & \nearrow & & \nearrow & & \nearrow & \\ (2,1) & & (2,2) & & (2,3) & & (2,4) & & \cdots\cdots \\ & \nearrow & & \nearrow & & \nearrow & & \nearrow & \\ (3,1) & & (3,2) & & (3,3) & & (3,4) & & \cdots\cdots \\ & \nearrow & & \nearrow & & \nearrow & & \nearrow & \\ (4,1) & & (4,2) & & (4,3) & & (4,4) & & \cdots\cdots \\ & \nearrow & & \nearrow & & \nearrow & & & \cdots\cdots \\ \vdots & & \vdots & & \vdots & & \vdots & & \cdots\cdots \\ & & & & & & & & \cdots\cdots \end{array}$$

In the above table first row consists of all ordered pairs of positive integers with first element 1. The second row consists of all ordered pairs of positive integers with first element 2 and so on. Also, in each row elements are arranged according to the increasing order of their second element. Then take the elements diagonal wise as shown in the diagram. Then elements of $\mathbb{N} \times \mathbb{N}$ can be written as an infinite sequence as below:

$$\{(1,1), (2,1), (1,2), (3,1), (2,2), (1,3), (4,1), (3,2), (2,3), \ldots\}$$

The key to listing the elements of $\mathbb{N} \times \mathbb{N}$ in a sequence is to first list the ordered pairs (p, q), with $p + q = 2$, followed by those with $p + q = 3$, followed by those with $p + q = 4$ and so on. The ordered pairs (p, q), with $p + q = k$ are arranged among themselves according to the increasing order of their second element. Thus the elements of $\mathbb{N} \times \mathbb{N}$ can be arranged as a sequence. Hence the set $\mathbb{N} \times \mathbb{N}$ is denumerable.

Theorem 1. *Suppose that S and T are sets and that $T \subseteq S$.*

(a) *If S is a countable set, then T is a countable set.*

(b) *If T is an uncountable set, then S is an uncountable set.*

Proof. (a) If S is a finite set, then from theorem 2 in § 4, it follows that T is finite and hence is countable. Now let S be denumerable. Then elements of S can be arranged as a sequence of distinct elements.

Let $$S = \{x_1, x_2, x_3, \ldots, x_n, \ldots\}.$$

Now construct a sequence $\{n_k\}$ as follows:

Let n_1 be the smallest postive integer such that $x_{n_1} \in T$. Let n_2 be the smallest integer greater than n_1 such that $x_{n_2} \in T$. Having chosen n_3, n_4, ..., n_{k-1} in a similar manner, let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in T$. This process can be carried indefinitely.

Putting $f(k) = x_{n_k}$ $(k = 1, 2, 3, \ldots..)$, we obtain a 1 – 1 correspondence between $\mathbb{N}$ and T. Hence T is denumerable.