Third Semester

LOGARITHMIC AND EXPONENTIAL FUNCTIONS

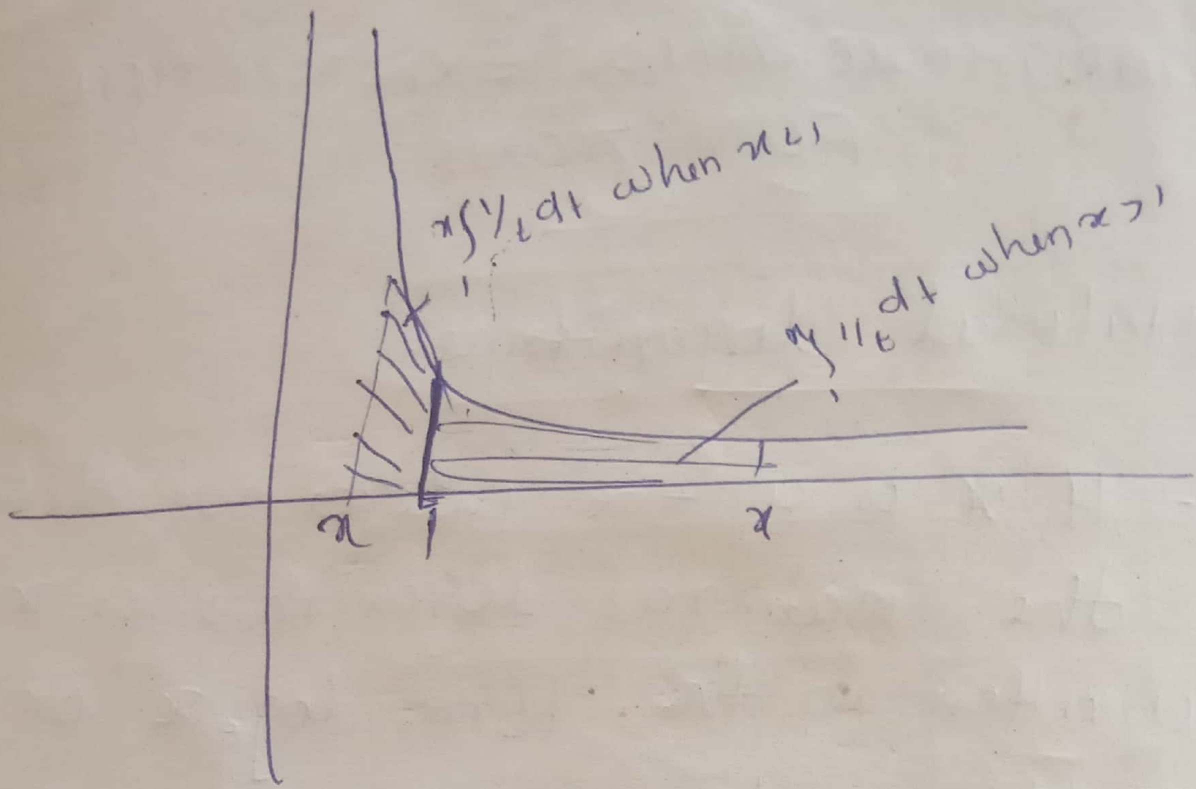
1. Natural logarithms.

defn:- If x is a +ve integer we define the natural logarithm of x written in the form $\log_e x$ or $\ln x$. as the value of the integral of the function $y = 1/t$ from $t=1$ to $t=x$.

eg:- ie $\ln x = \int_1^x 1/t \, dt$. $x > 0$.

eg $\ln 1 = \int_1^1 1/t \, dt = 0$ (since $\int_a^a f = 0$).

Graph of $\ln x$



The derivative of $y = \ln x$.

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0.$$

$$\frac{d}{dx}(\ln x) = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt.$$

$$= \frac{1}{x}$$ [By Fundamental theorem of calculus part 1].

Properties of natural logarithms.

1. Product rule.

$\ln(ax) = \ln a + \ln x$, where $a > 0$ & $x > 0$.

Proof:- $\frac{d}{dx}(\ln ax) = \frac{1}{ax} \times a = 1/x.$ $\left[\because \frac{d}{dx}(\ln x) = 1/x.\right.$

also $\frac{d}{dx}(\ln x) = 1/x.$

$\Rightarrow \ln ax = \ln x + c.$ — (1)

To evaluate c put $x = 1$ in (1)

Then $\ln a = \ln 1 + c$

$= 0 + c$

$\Rightarrow \ln a = c.$

Then substitute this in (1) we get.

$$\boxed{\ln ax = \ln x + \ln a}.$$

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Quotient Rule

$\ln(x/a) = \ln x - \ln a$ where $a>0$ & $x>0$

Proof: In the product rule proved above put $x = 1/a$ then we get

$$\ln 1 = \ln(a \cdot 1/a)$$
$$= \ln a + \ln 1/a.$$

i.e. $$0 = \ln a + \ln 1/a.$$

$$\ln 1/a = -\ln a.$$

Then $$\ln(x/a) = \ln(x \cdot 1/a)$$
$$= \ln x + \ln 1/a$$
$$= \ln x - \ln a$$

3. Power rule

$\ln x^n = n \ln x$ where $x > 0$ and n is a rational number.

$$\frac{d}{dx}(\ln x^n) = \frac{1}{x^n} \cdot n x^{n-1}$$

$$= \frac{n}{x}$$

$$= n \times 1/x$$

$$= n \, d/dx \ln x$$

$$= \frac{d}{dx}(n \ln x)$$

Also $\frac{d}{dx}(n \ln x) = n \cdot 1/x$.

$$\boxed{\therefore \ln x^n = n \ln x}$$

Problems

using properties of logarithms.

Simplify.

1) $\ln(3x^2-9x)+\ln(1/3x)$

$$= \ln\left(\frac{3x^2-9x}{3x}\right) \quad \left[\because \ln a + \ln b = \ln a/b\right]$$

$$= \ln \frac{\cancel{3x}(x-3)}{\cancel{3x}}$$

$$= \ln x-3$$

2. $\ln(8x+4)-2\ln 2$.

3 $\frac{1}{2}\ln(4t^2)-\ln 2$.

4 Express the following logarithms in terms of $\ln 5$ and $\ln 7$.

a) $\ln 1/125$

Ans:. $\ln\left(\frac{1}{125}\right) = \ln\left(\frac{1}{5^3}\right)$

$= \ln 1 - \ln 5^3$

$= 0 - 3\ln 5$ $\Big($ since $\ln(a/b) = \ln a - \ln b$.
$\ln 1 = 0$.
$\ln x^n = n \ln x \Big)$.

b). $\ln(4/9)$ in terms of $\ln 2$ & $\ln 3$.

c) $\ln(1225)$ in terms of $\ln 5$ & $\ln 7$.

The Exponential Function.

Defn:- For every real number x, the number e^x is defined to be the number whose natural logarithm is x.

i.e $e^x = \ln^{-1} x$.

$$y = e^x \iff x = \ln y.$$

Remark:- Since $\ln x$ and e^x are inverse of one another,

Laws of exponents

Law 1. For any two real numbers x_1 and x_2:

$$e^{x_1} e^{x_2} = e^{x_1 + x_2}.$$

Proof :- Let $y_1 = e^{x_1}$ and $y_2 = e^{x_2}$.

$\ln y_1 = \ln e^{x_1}$ and $y_2 = e^{x_2}$

$\Rightarrow \ln y_1 = x_1$ & $\Rightarrow \ln y_2 = \ln e^{x_2}$

& $\Rightarrow y_2 = x_2$.

hence $x_1 + x_2 = \ln y_1 + \ln y_2$.

$= \ln y_1 y_2$.

But $\ln y_1 y_2 = x_1 + x_2$.

$\Rightarrow y_1 y_2 = e^{x_1 + x_2}$.

$$\boxed{e^{x_1} e^{x_2} = e^{x_1 + x_2}.}$$

Law 2 :- For any real number x.

$e^{-x} = 1/e^{x}$.

Proof :- Let $y = e^{x}$.

Now $y = e^x \Longrightarrow \ln y = x$.

hence $-x = -\ln y$

$= \ln y^{-1}$ $\left(\because \ln x^n = n \ln x \quad \ln y^{-1} = -1 \times \ln y\right)$

$= \ln 1/y$ $\left(y^{-1} = 1/y\right)$.

i.e. $-x = \ln(1/y)$

$\Longrightarrow e^{-x} = 1/y$

$$\boxed{\Longrightarrow e^{-x} = 1/e^x.}$$

<u>law 3</u> For any two real numbers x_1 and x_2

$$\left(e^{x_1}\right)^{x_2} = e^{x_1 x_2} = \left(e^{x_2}\right)^{x_1}$$

Proof:- Let $y = e^{x_1}$

$$\Rightarrow \ln y = \ln e^{x_1}$$

$$\Rightarrow \ln y = x_1.$$

hence $x_1 x_2 = x_2 \ln y = \ln y^{x_2}$.

$$\cancel{\Rightarrow e^{x_1 x_2} = y}$$

taking exponential

$$e^{x_1 x_2} = e^{\ln y^{x_2}}.$$

ie $e^{x_1 x_2} = y^{x_2} = (e^{x_1})^{x_2}.$

Similarly we can prove that

$$(e^{x_2})^{x_1} = e^{x_1 x_2}.$$

<u>Remark</u> : $e^0 = 1$ and $e^1 = e$.

Derivative and integral of e^x

$\frac{d}{dx}(e^x) = ?$

Let $y = e^x$. taking logarithm on both sides.

$$\ln y = \ln e^x$$

$$\Rightarrow \ln y = x.$$

Differentiating both sides of the equation w.r to x.

$$\frac{1}{y}\frac{dy}{dx} = 1$$

[$\frac{d}{dy}(\ln y) = 1/y$.
ഇവിടെ w.r to x ആയതു കൊണ്ട് $\frac{d}{dx}(\ln y) = 1/y \cdot \frac{dy}{dx}$ ആകും].

$$\Rightarrow \frac{dy}{dx} = y$$

$$\Rightarrow \frac{dy}{dx} = e^x$$

ie $\frac{d}{dx}(e^x) = e^x$ //

Also $\int e^u \, du = e^u + c$.

Problems :-

(1) simplify $\ln(\ln e^e)$. — (1)

$(\ln e^e) = e$.

(1) becomes $\ln(e) = 1$

(~~log~~ ln ഉം e ഉം inverse to each other. അതുകൊണ്ട് $\ln e = 1$)

$$\boxed{\therefore \ln(\ln e^e) = 1}$$

(2) simplify $e^{(\ln x - \ln y)}$

$$e^{\ln x - \ln y}$$

$$= e^{\ln(x/y)}$$

$$= x/y.$$

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Problem (3) solve for y:-

$$e^{x^2} \cdot e^{2x+1} = e^{y}.$$

Soln :- we have $e^{x^2} e^{2x+1} = e^{y}$.

~~Taking~~ ie $e^{x^2+2x+1} = e^{y}$ $\left[\because e^{x_1} e^{x_2} = e^{x_1+x_2}\right]$

Taking logarithm on both sides we get

$$\ln e^{x^2+2x+1} = \ln e^{y}.$$

$\Rightarrow x^2+2x+1 = y$ (Since ln and e are inverse to each other).

$\Rightarrow (x+1)^2 = y$

(4) Find $\frac{dy}{d\theta}$ if $y = \ln\left(\frac{e^{\theta}}{1+e^{\theta}}\right)$.

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soln :- $\frac{d}{d\theta}\left(\ln \frac{e^\theta}{1+e^\theta}\right) = \frac{1}{e^\theta/1+e^\theta} \cdot \frac{d}{d\theta}\left(\frac{e^\theta}{1+e^\theta}\right).$

[$\because \frac{d}{dx} \ln x = 1/x$ & function of function rule)

$$= \frac{1+e^\theta}{e^\theta} \cdot \frac{(1+e^\theta)\times e^\theta - e^\theta . e^\theta}{(1+e^\theta)^2}.$$

$$= \frac{1+e^\theta}{e^\theta} \cdot \frac{e^\theta + \cancel{e^{2\theta}} - \cancel{e^{2\theta}}}{(1+e^\theta)^2}$$

$$= \frac{\cancel{1+e^\theta}}{e^\theta} \cdot \frac{e^\theta}{(1+e^\theta)^2} \quad \cancel{(1+e^\theta)(1+e^\theta)}$$

$$= \frac{1}{1+e^\theta}$$

5) Evaluate $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ — (1)

soln :- using substitution method to solve this

put ~~$\sqrt{x}$~~ $u=\sqrt{x}$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$2\,du = \frac{dx}{\sqrt{x}}$$ substitute

these in (1) we get

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int_1^2 e^u \times 2\,du.$$

when $x=1$ $u=\sqrt{x}$ $\Rightarrow u=1$

when $x=4$ $u=\sqrt{4}$ $=2$

$$= 2\left[e^u\right]_1^2$$

$$= 2(e^2 - e^1)$$

$$= 2(e^2 - e)$$

The Function a^x.

We define the function

by $a^x = e^{\ln a^x} = e^{x \ln a}$.

Law of exponents.

i) $a^x a^y = a^{x+y}$.

ii) $1/a^x = a^{-x}$.

iii) $a^x/a^y = a^{x-y}$.

iv) $(a^x)^y = a^{xy} = (a^y)^x$.

defn :- If $a>0$ and x is any real number, then $a^x = e^{x \ln a}$.

Derivative of a^x.

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x \ln a})$$

$$= e^{x \ln a} \cdot \frac{d}{dx}(x \ln a)$$

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$= e^{x \ln a} \cdot \ln a$

$= e^{\ln a^x} \ln a$

$= a^x \ln a.$

ie $\frac{d}{dx}(a^x) = a^x \ln a.$

Integral of a^x..

$$\boxed{\int a^x dx = \frac{a^x}{\ln a} + c}$$

Problems:- Differentiating the following functions with respect to x.

$y = x 2^{x^2}.$

$\frac{dy}{dx} = x \cdot \frac{d}{dx}(2^{x^2}) + 2^{x^2} \frac{d}{dx}(x)$ (

~~ex~~ (By product rule for Differentiation

$$= x \times 2^{x^2} \ln 2 \frac{d}{dx}(x^2) + 2^{x^2} \times 1$$

$\left(\because \frac{d}{dx}(a^x) = a^x \ln a. \right.$

here $x = x^2$. $)$.

$$= 2^{x^2}\left[x \ln 2 \times 2x + 1\right]$$

$$= 2^{x^2}\left[2 \ln 2\, x^2 + 1\right]$$

2) $y = 3^{\sin x}$.

$$\frac{dy}{dx} = 3^{\sin x} \ln 3 \frac{d}{dx}(\sin x)$$

$$= 3^{\sin x} \ln 3 (\cos x).$$

Evaluate the following integral.

i) $\int 2^{\sin 3t} \cos 3t \, dt$.

Put $\sin 3t = x$.

$dx = 3 \cos 3t \, dt$.

Then $\int 2^{x} \frac{1}{3} \, dx$.

$= \frac{2^{x}}{3 \ln 2} + c = \frac{2^{\sin 3t}}{3 \ln 2} + c$.

ii) $\int_{1}^{\sqrt{2}} x \cdot 2^{-x^2} \, dx$.

Put $x^2 = u$.

$2x \, dx = du$.

$\therefore \int_{1}^{\sqrt{2}} x 2^{-x^2} dx = \int 2^{-u} \frac{du}{2}$

$x = 1 \quad u = 1$

$x = \sqrt{2} \quad u = 2$.