part 1 Electrodynamics - I

chapter 1 Vector Calculus

1.1 Vector Algebra

1.1.1 Vector Operations

first ⟶ Scalars & Vectors

Scalars ↓ physical quantity which can be completely specified by magnitude. eg:- e potential, flux, energy, capacitance, resistance, mass volume, temp. etc

Vectors ↓ which can be completely specified by magnitude and direction. eg:- force, displacement, acceleration, dipole moment, mag. vector potential etc

if A is a vector, then it is represented as $\vec{A}$ A A

" unit vector " $\hat{A}$, $\vec{A} = A\hat{A}$

$$\hat{A} = \frac{\vec{A}}{|A|}$$ $|A| \longrightarrow$ magnitude of the vector.

arrow headed pointing straight line. { $\xrightarrow{A}$, $\xleftarrow[-A]{}$ } direction changed, but same magnitude ∴ -ve sign

This is the geometrical representation of a vector.

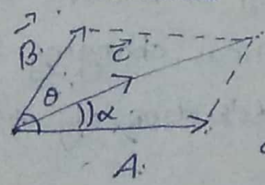
- **Adition of Vectors.**

- equal vectors ⟶ if two vectors have equal magnitude and same direction

coplanar vectors ⟶ if two or more vectors are in same plane.

Vectors are added based on 2 rules

i) parallelogram rule.



If two vectors $\vec{A}$ and $\vec{B}$ drawn from the same point, form the adjacent sides of a parallelogram, then the diagonal vector $\vec{C}$ will be the resultant of the two vectors.

$$\vec{C} = \vec{A} + \vec{B}$$

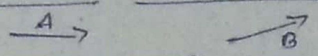
Magnitude of resultant vector, $|C| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

Direction of resultant vector, $\vec{C}$

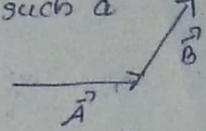
$$\tan\alpha = \frac{B\sin\theta}{A + B\cos\theta}$$

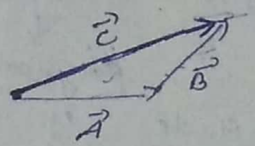
(2)

ii) Triangular rule or head to tail rule



If two vectors $\vec{A}$ and $\vec{B}$ are positioned in such a way that head of $\vec{A}$ connects the tail of $\vec{B}$ (forming two sides of a triangle in order), then their sum will be the vector drawn from the tail of $\vec{A}$ to the head of $\vec{B}$ (forming the third side of triangle in reverse order)





Vector addition obeys commutative law

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

and associative law $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$

To substract a vector, add its opposite

so, $A - B = A + (-B)$, $(-B)$ is -ve of $\vec{B}$

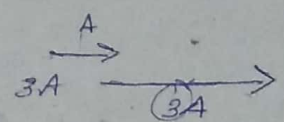
$\vec{C} \Rightarrow$ the resultant of vector

$$\vec{C} = A + -B$$

$$|\vec{C}| = \sqrt{A^2 + B^2 - 2AB\cos\gamma}$$

Product of Vectors

i) Multiplication by a scalar



Let α be the scalar

$\vec{A}$ be the vector

~~Then~~ Now, multiply this vector $\vec{A}$, with the positive scalar α, then magnitude of the $\vec{A}$ (vector) will be changed by α times

statement

Multiplication of a vector by a positive scalar α changes its magnitude by α times without changing the direction of vector. If α is -ve, direction of resultant vector will be ~~chan~~ reversed

$$\alpha\vec{A} = \alpha A\hat{A}$$

$$-\alpha\vec{A} = \alpha A(-\hat{A})$$

$\vec{A} = A\hat{A}$ → representation

$\hat{A}$ → direction, A → magnitude

Scalar multiplication obeys distributive law

$$\alpha(\vec{A} + \vec{B}) = \alpha\vec{A} + \alpha\vec{B}$$

Multiplication of 2 vectors are done by 2 different vectors.

ii) Dot product or Scalar product of two vectors

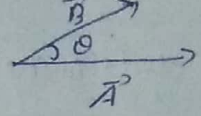
The dot product of two vectors is defined by

$\vec{A}\cdot\vec{B} = AB\cos\theta$, (the result will be a scalar)

where, A and B are magnitudes of respective vectors

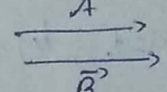
θ is the smaller angle b/w $\vec{A}$ and $\vec{B}$

(always less than 180°)



- Dot product is commutative i.e, $A\cdot B = B\cdot A$.
- Dot product is distributive i.e, $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$.

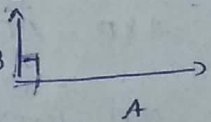
- suppose A and B are parallel. i.e,



$\theta = 0^\circ$

$\therefore \vec{A}\cdot\vec{B} = AB\cos 0 = AB$. $\Rightarrow \underline{\underline{\vec{A}\cdot\vec{B} = AB}}$

- if A and B are $\perp$r i.e



$\vec{A}\cdot\vec{B} = AB\cos 90 = AB\times 0$

$\underline{\underline{\vec{A}\cdot\vec{B} = 0}}$

<u>class work</u> (Must do this and submit now).

Let $C = A - B$ and calculate the dot product of C with itself. (i.e, $C\cdot C$) Then you will get the law of cosine for of triangle

(Do this and submit as assignment)

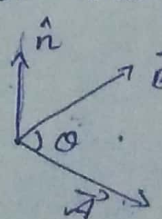
iii) Cross product of two vectors. or vector product

cross product of two vectors are defined as

$\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$ (result will be a vector)

A and B are the magnitudes of respective vector

θ is the smaller angle b/w them.



How to give direction.

Direction of $\vec{A}\times\vec{B}$ is given by Right hand rule (direction of thumb when the fingers are rotated from $\vec{A}$ to $\vec{B}$)

~~Also the resultant vector will~~

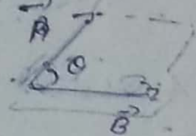
So, the vector or cross product of 2 vectors $\vec{A}$ & $\vec{B}$ is a vector in the direction ⊥r to the plane containing $\vec{A}$ and $\vec{B}$.

$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ does not obey commutative law

distributive law $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

$\vec{A} \times (B \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$ → not associative.

Also $|A \times B|$ ⟹ the area of the parallelogram generated by $\vec{A}$ and $\vec{B}$



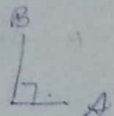
- Supposed two vectors are parallel,

$\vec{A} \times \vec{B} = AB \sin 0$

$= 0$

$A \times B = AB \sin \theta$

- Suppose two vectors are ⊥r



$\vec{A} \times \vec{B} = AB \sin 90$

$= AB$

- $A \times A = 0$

? Using the General eq. for dot product and cross product and with appropriate diagrams, show that dot product and cross product are distributive

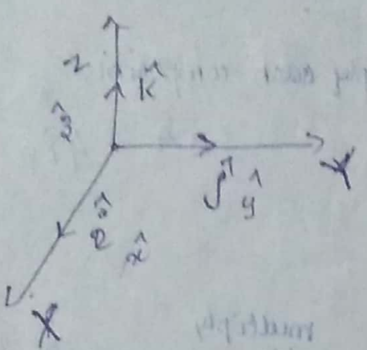
a) when the three vectors are co-planar

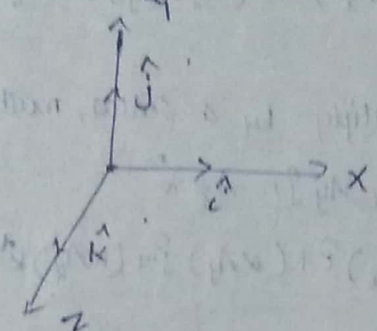
b) in the general case.

part 2 (5)

1.1.2 Vector Algebra : Component Form

There are many types of co-ordinate system, in ~~par~~ practice, ~~we~~ it is easier to use cartesian co-ordinate x, y, z to study the vector components.



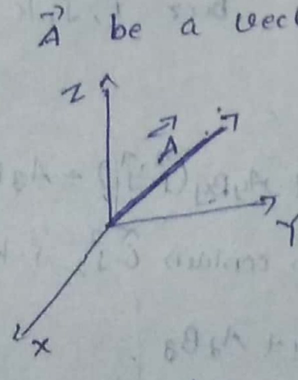


Right hand screw rule is used to find the x, y, z in a Cartesian co-ordinate system :

Draw X and Y axis as we like, the rotate ring hand screw from x to Y, then the direction of advancement of screw gives Z-axis

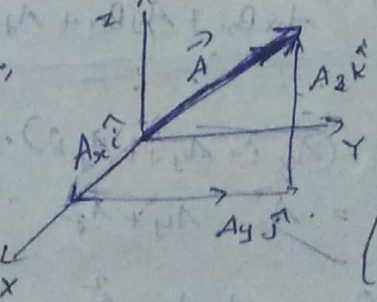
$\hat{i}, \hat{j}, \hat{k}$ or $\hat{x}, \hat{y}, \hat{z}$ are the unit vectors along the direction axis X, Y, Z respectively

Let $\vec{A}$ be a vector in a cartesian co-ordinate system,



• Look at the geometry we can split this vector $\vec{A}$ into 3 components along x, Y and z direction

i.e,



Actually A_x, A_y, A_z are the projections of A along three axis

$$\therefore \vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

or

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

These A_x, A_y and A_z are known as components or rectangular components of the vector $\vec{A}$.

- To add vectors, add like components

ie, let $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

$\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$
$\vec{B} = 6\hat{i} + 7\hat{j} + 8\hat{k}$
$\vec{A}+\vec{B} = \underline{8\hat{i} + 10\hat{j} + 12\hat{k}}$

$$\vec{A}+\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) + (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$\underline{\vec{A}+\vec{B} = (A_x+B_x)\hat{i} + (A_y+B_y)\hat{j} + (A_z+B_z)\hat{k}}$$

- To multiply by a scalar, multiply each component

$A = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$\underline{\alpha\vec{A} = (\alpha A_x)\hat{i} + (\alpha A_y)\hat{j} + (\alpha A_z)\hat{k}}$

$\alpha = 2$

$2\vec{A} = \underline{4\hat{i} + 6\hat{j} + 8\hat{k}}$

- To calculate the dot product, multiply like components & add

$$\vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

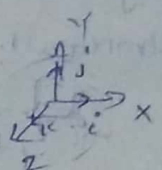
Multiply each component of $\vec{A}$ with $\vec{B}$

for eg. $\underline{A_x\hat{i}\cdot B_x\hat{i} + A_x\hat{i}\cdot B_y\hat{j} + A_x\hat{i}\cdot B_z\hat{k}}$

$\downarrow$

$A_xB_x(\hat{i}\cdot\hat{i}) + A_xB_y(\hat{i}\cdot\hat{j}) + A_xB_z(\hat{i}\cdot\hat{k})$

$\hat{i}\cdot\hat{j} = ij\cos\theta$



$\hat{i}\cdot\hat{i} = ii\cos\theta$

$\hat{i}\cdot\hat{i} = 1 \quad \hat{i}\cdot\hat{j} = 0$
$\hat{j}\cdot\hat{j} = 1 \quad \hat{i}\cdot\hat{k} = 0$ becz $\hat{i}, \hat{j}, \hat{k}$ are mutually $\perp r$.
$\hat{k}\cdot\hat{k} = 1 \quad \hat{j}\cdot\hat{k} = 0$

$\therefore \vec{A}\cdot\vec{B} = A_xB_x(\hat{i}\cdot\hat{i}) + A_yB_y(\hat{j}\cdot\hat{j}) + A_zB_z(\hat{k}\cdot\hat{k})$ only exists

(all other terms which contain $\hat{i}\cdot\hat{j}$, $\hat{j}\cdot\hat{k}$ etc all other becm zero)

$$\boxed{\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z}$$

$$\boxed{\begin{aligned}\vec{A}\cdot\hat{i} &= A_x\\ \vec{A}\cdot\hat{j} &= A_y\\ \vec{A}\cdot\hat{k} &= A_z\end{aligned}}$$

$\vec{A}\cdot\vec{A} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(A_x\hat{i} + A_y\hat{j} + A_z\hat{k})$

$= A_x^2 + A_y^2 + A_z^2$

$AA\cos 0$ $\quad \therefore A\cdot A = A^2$

$\cos 0$

$\therefore A^2 = A_x^2 + A_y^2 + A_z^2$

$\underline{A = \sqrt{A_x^2 + A_y^2 + A_z^2}}$

• To calculate the cross product, form the determinant whose first row is $\hat{x}, \hat{y}, \hat{z}$, whose second row is $\vec{A}$ (component form) and third row is $\vec{B}$ (component form)

Now, you do this

$$\cancel{\vec{A}\times\vec{B} = (A_x \hat{x} + A_y \hat{y} + A_z \hat{z}) \times A_y \hat{y} + A_z \hat{z} + A}$$

$$\vec{A}\times\vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$

$\hat{i}\times\hat{i} = ii \cos 90$ $\sin\theta\, \hat{n}$

hint : $\cancel{\hat{x}\times\hat{x}}$ $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$

$\hat{i}\times\hat{j} = \hat{k}$ $\quad \hat{j}\times\hat{i} = -\hat{k}$

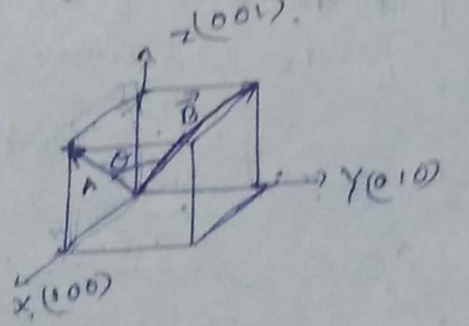
$\hat{j}\times\hat{k} = \hat{i}$ $\quad \hat{k}\times\hat{j} = -\hat{i}$

$\hat{k}\times\hat{i} = \hat{j}$ $\quad \hat{i}\times\hat{k} = -\hat{j}$

(k) $\quad i, k, j$ follow cyclic direction

$$\therefore \vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

? Find the angle b/w the face diagonals of a cube.



$A = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$= 1\hat{i} + 0\hat{j} + 1\hat{k}$

$B = 0\hat{i} + 1\hat{j} + 1\hat{k}$

we have, $A\cdot B = AB\cos\theta$

we need to find θ

for that we need $A\cdot B$

find $A\cdot B$ in component form

$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$

$= 1\cdot 0 + 0\cdot 1 + 1\cdot 1$

$= 1$

(look into figure) and check whether $A_x, B_x, A_y, B_y, A_z, B_z$ components are present or not (1 or 0)

Now we need $AB \Rightarrow |A|\,|B|$

magnitude of A

Where $A = \sqrt{A\cdot A} = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{1^2 + 0 + 1^2} = \sqrt{2}$

lly $B = \sqrt{0 + 1^2 + 1^2} = \sqrt{2}$

$1 = \sqrt{2}\sqrt{2}\cos\theta$

$1 = 2\cos\theta$

$\Rightarrow \cos\theta = 1/2$

$\theta = \cos^{-1} 1/2$

$\Rightarrow \underline{\underline{\theta = 60^\circ}}$

Scanned with CamScanner

Assignment. (class work)

Find the angle b/w the body diagonals of a cube

(Do this and submit now)

1.1.3 Tripple product: $\vec{A} \times \vec{B}$ - $\vec{C} \times \vec{D}$ ($\vec{C} \cdot \vec{D}$)

$\vec{A} \cdot \vec{B} = C$

~~Since~~ we have seen that result of dot product b/w 2 vectors was a scalar but cross product of two vectors gives a vector itself. So ~~we take~~ this vector can be dotted or crossed with a third vector to form a tripple product

i) scalar tripple product :- $\vec{A} \cdot (\vec{B} \times \vec{C})$

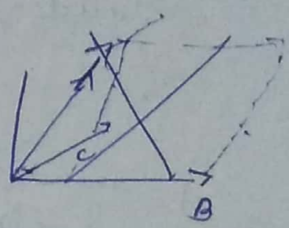
~~A. BxC~~

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) \longrightarrow (1)$$

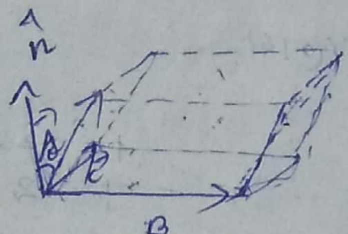
Geometrically $|\vec{A} \cdot (\vec{B} \times \vec{C})|$ is the volume of the parallelepiped generated by A, B and C

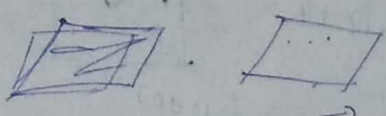
$|\vec{B} \times \vec{C}|$ is the area of the base

$|A \cos\theta|$ is the attitude



~~figure~~





$\vec{A} = 2\hat{i} + 3\hat{j} + 5\hat{k}$

$\vec{B} = 4\hat{i} + 6\hat{j} + 8\hat{k}$

$\vec{C} = 7\hat{i} + 9\hat{j} + 10\hat{k}$

eq. (1) has cyclic permutation

In component form :-

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

find $\vec{B} \cdot (\vec{C} \times \vec{A})$ and $C \cdot (\vec{A} \times \vec{B})$

do it now class work.

ii) Vector tripple product: $A\times(\vec{B}\times\vec{C})$

$A = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

$\vec{A}\times(\vec{B}\times\vec{C})$ is simply known as BAC - CAB rule

__class work__ (do this and submit today itself)

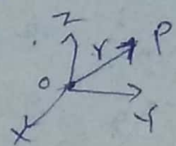
i) Prove BAC-CAB rule by writing out both sides in component form

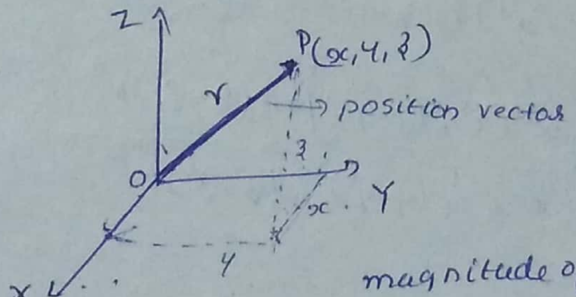
ii) $[\vec{A}\times(\vec{B}\times\vec{C})] + [\vec{B}\times(\vec{C}\times\vec{A})] + [\vec{C}\times(\vec{A}\times\vec{B})] = 0$. prove this.

under what conditions does $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\times\vec{C}$

1.1.4 position, Displacement, and Seperation Vectors.

The location of a point in 3-D can be described by listing its Cartesian co-ordinates (x, y, z). The vector to that point from the origin is called __position vector__.





$$\boxed{\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}}$$

magnitude of $r = \sqrt{x^2+y^2+z^2}$

$\vec{r}\cdot\vec{r} = r^2$

This magnitude is the distance from origin

$\vec{r}\cdot\vec{r} = x^2+y^2+z^2$

$\vec{r} = r\hat{r}$ → unit vector; r → magnitude

~~$r\cos\theta$~~ $r^2 =$

$$\therefore \hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$ gives the unit vector pointing radially outward

The infinitesimal __displacement vector__ from (x, y, z) to $(x+dx, y+dy, z+dz)$ is

$(x+dx, y+dy, z+dz)$

(x,y,z)

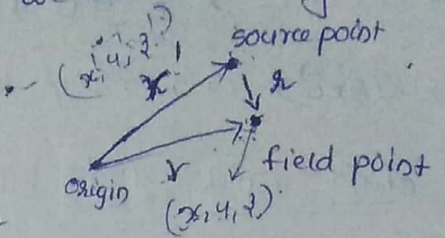
$$\boxed{dl = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}}$$

can use 'dr' also instead of dl

$$dr = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

In E-D, we will see many problems involving 2 points

1. a source point '$\vec{r}'$', where the charge is located
2. a field point '$\vec{r}$', at which we are calculating electric or mag. field.



The short seperation b/w source point & field point is known as seperation vector denoted by '$\vec{\mathscr{r}}$'

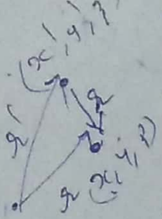
$$\vec{\mathscr{r}} = r - r'$$

$$\mathscr{r} = |r - r'|$$

$$\vec{\mathscr{r}} = \mathscr{r}\,\hat{\mathscr{r}}$$

$$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}} = \frac{r - r'}{|r - r'|}$$ → unit vector in the direction from $\vec{r}'$ to $\vec{r}$

In cartesian co-ordinates



$$\vec{\mathscr{r}} = (x - x')\hat{i} + (y - y')\hat{j} + (z - z')\hat{k}$$

$$\mathscr{r} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

hint $\left(\mathscr{r}^2 = \vec{\mathscr{r}} \cdot \vec{\mathscr{r}}\right)$ (do this)

$$\hat{\mathscr{r}} = \frac{(x - x')\hat{i} + (y - y')\hat{j} + (z - z')\hat{k}}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}$$

1.1.5 How Vectors Transforms

Actually what is a vector ?. is the definition a quantity with magnitude & direction satisfactory ?. let's see
We have studied abt component form. So we will say that vector is anything that has 3 components that combine properly under addition. Now look this $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

We have a barrel of fruits that contains N_x mango, N_y apples and N_z bananas. Then is $N = N_x\hat{i} + N_y\hat{j} + N_z\hat{k}$ a vector ? It has 3 components.
If we add another barrel of fruits M_x mango, M_y apple, M_z banana $M = M_x\hat{i} + M_y\hat{j} + M_z\hat{k}$
then $(N_x + M_x)$ mango, $(N_y + M_y)$ apple, $(N_z + M_z)$ bananas. It ~~add~~ does add like a vector. But we know, it is not a vector. why ? It does not have direction
we will say like this, what is wrong with this ?

Actually the answer is that

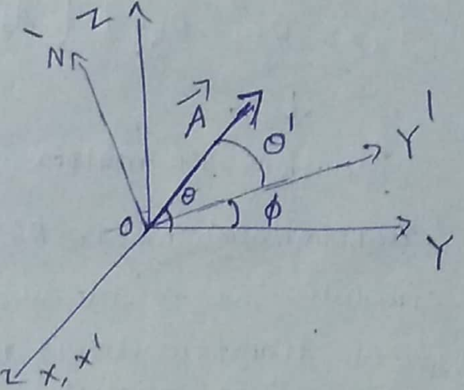
N does not transform properly when you change co-ordinate

(so vectors will transform when we change the co-ordinate.)
(or must)

Let's see how this happens.

There is specific geometrical transformation law for converting vector components in one set of co-ordinates into another. But scalar quantities are invariant.

Consider a system with X, Y, Z co-ordinate which is rotated ~~abt~~ at an angle ϕ ~~with~~ abt common X-axis



$$\theta' = \theta - \phi$$

Now let's see how components of A changes

w.r.t XYZ $\quad A_y = A\cos\theta$, $A_z = A\sin\theta$ } before rotating.

w.r.t X'Y'Z' i.e. after rotating.

$$A_y' = A\cos\theta' = A\cos(\theta-\phi) = A(\cos\theta\cos\phi + \sin\theta\sin\phi)$$

$$= \underline{A\cos\theta}\cos\phi + \underline{A\sin\theta}\sin\phi$$

$$A_y' = A_y\cos\phi + A_z\sin\phi$$

lly $\quad A_z' = A\sin\theta' = A\sin(\theta-\phi) = A(\sin\theta\cos\phi - \cos\theta\sin\phi)$

$$= \underline{A\sin\theta}\cos\phi - \underline{A\cos\theta}\sin\phi$$

$$= A_z\cos\phi - A_y\sin\phi$$

$$A_z' = -\sin\phi . A_y + \cos\phi\, A_z$$

$$A_y' = \cos\phi\, A_y + \sin\phi\, A_z$$

$$\begin{pmatrix} A_z' \\ A_y' \end{pmatrix} = \begin{pmatrix} -\sin\phi & \cos\phi \\ \cos\phi & \sin\phi \end{pmatrix} \begin{array}{c} A_y \\ A_z \end{array}$$

$$A_y' = \cos\phi\, A_y + \sin\phi\, A_z$$

$$A_z' = -\sin\phi\, A_y + \cos\phi\, A_z$$

In matrix form

$$\begin{pmatrix} A_y' \\ A_z' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_y \\ A_z \end{pmatrix}$$

In general

$$\begin{pmatrix} A_y' \\ A_z' \end{pmatrix} = \begin{pmatrix} R_{yy} & R_{yz} \\ R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_y \\ A_z \end{pmatrix}$$

($R \Rightarrow$ rotation).

$\cancel{\vec{A}' = A_y + A_z}$

In 3D

$$\begin{pmatrix} A_x' \\ A_y' \\ A_z' \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$\downarrow$

Transformation matrix R

The elements of Transformation matrix 'R' is determined depending upon relative orientation of co-ordinate axes.

This type of transformation is the most important characteristic of any vector quantity.

In compact form

$$A_i' = \sum_{j=1}^{3} R_{ij} A_j$$

j = 1, 2, 3. respresents x, y, z axes respectively

A scalar quantity is considered as tensor of zero rank

A vector " " " rank 1

Part 5

1.2 Differential calculus

1.2.1 Ordinary Derivatives

Question: Suppose we have a function of one variable : $f(x)$
what does the derivative $\frac{df}{dx}$, do for us?

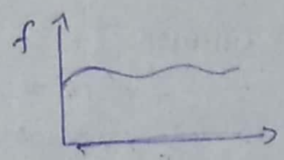
Answer: It tells us how rapidly the function $f(x)$ varies when we change the argument x by a tiny amount, dx;

$$df = \left(\frac{df}{dx}\right) dx \longrightarrow (1)$$

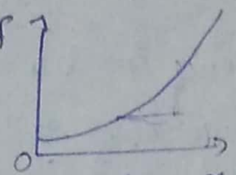
In words: If we change x by an amount dx, then f changes by an amount df; the derivative is a proportionality factor.

$f = f(x)$

Look



(a) x



(b). x

- Here function varies slowly with x.
- Small derivative

- Here f increases rapidly with x
- and derivative is large, as you more away from $x = 0$

Geometrical interpretation: The derivative $\frac{df}{dx}$ is the slope of the graph of f versus x.

Some useful derivatives

i) $\frac{d}{dx}(\vec{A}+\vec{B}) = \frac{d\vec{A}}{dx} + \frac{d\vec{B}}{dx}$

ii) $\frac{d}{dx}(\vec{A}\cdot\vec{B}) = \vec{A}\cdot\frac{d\vec{B}}{dx} + \frac{d\vec{A}}{dx}\cdot\vec{B}$

iii) $\frac{d}{dx}(\vec{A}\times\vec{B}) = \vec{A}\times\frac{d\vec{B}}{dx} + \frac{d\vec{A}}{dx}\times\vec{B}$

- A scalar or a vector function is said to be differentiable of the order 'n', if n^{th} derivative exists

eg:- $y = \frac{1}{2}kx^2$

$\frac{dy}{dx} = kx$ $\frac{d^2y}{dx^2} = k$; $\frac{d^3y}{dx^3} = 0$ $\therefore$ This function is differentiable of the order 2.

- Scalar field :- It is a region of space containing the points abt which the values of a Scalar function are continuously distributed.

- Vector field :- It is a region of space containing the points abt which the value of a vector point function are continuously distributed.

1.2.2.

Gradient (∇)

Suppose, now that we have a function of 3 variables - say - temperature $T(x, y, z)$ (consider cartesian co-ordinate system T at origin starting point)

We want to generalize the meaning of "derivative" to function like T, which depends not on one but three variable

(at first we have seen that f is a f^n of x only ie, 1 variable
Here T is a function of 3 variable)

Now a derivative is supposed to tell us how fast the function varies, if we move a little distance. But this time the situation is more complicated, because it depends on what direction we move: If we go straigh up, then the temp. will probably ↑ fairly rapidly, but if move horizontally, it may not change much at all. In fact, the question "How fast does T vary"? has an infinite answers.

We have a Theorem on Partial derivative states that

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz \cdots \longrightarrow (2)$$

This tells us how T changes when we alter all 3 variables by very small amounts dx, dy, dz. (we use 3-D co-ordinates only 3 direction is choosen)

$$\cancel{dT = \left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}\right)}.$$

This $\frac{\partial T}{\partial x}$, $\frac{\partial T}{\partial y}$, $\frac{\partial T}{\partial z}$ are the partial derivatives of T along 3 directions (ie, rate of change of T in all 3 directions)

Let the magnitude of displacement vector is

$$dl = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

Then $dT = \left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}\right) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}) \longrightarrow (3)$

$$dT = \nabla T \cdot dl \longrightarrow (4)$$

where, $\boxed{\nabla T = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}}$ is the gradient of T.

∇T is a vector quantity with 3 components

↓

known as Generalised derivative

Actually eq. (4) is the 3D version of eq. (1)

Geometrical interpretation of the Gradient: Like any vector, the gradient has magnitude and direction. To determine its geometrical meaning.

$$\boxed{dT = \nabla T \cdot dl}$$

$$= |\nabla T||dl| \cos\theta$$

$\theta \Longrightarrow$ angle b/w ∇T and dl.

Let's fix the magnitude of $|dl|$ and search around in various direction (ie, vary the angle θ), then, the maximum change in T occurs when $\theta = 0°$ ie, dT is maxi when $\nabla T \cdot dl = \nabla T\, dl \cos\theta$

$\cos\theta = 1$

$\therefore \nabla T \cdot dl = \nabla T dl$.

ie, for a fixed distance $|dl|$, dT is greatest when l move in the same direction as ∇T

Thus, The gradient ∇T points in the direction of maximum increase of the function T.

More over.

The magnitude $|\nabla T|$ gives the slope along this maximum direction.

If ∇T and dl are ⊥r to each other, (angle $\theta = 90°$) then slope = zero.

What is the meaning for gradient to vanish? of ~~$\nabla T = 0$ at (x,y,z)~~.

If $\nabla T = 0$ (at x, y, z) then $dT = 0$ for small displacements abt (x,y,z). This is then stationary point of the function $T(x,y,z)$. It could be

a maximum, minimum, a saddle point
(mini-max point)

submit   worry

If you want to locate extrema of a three variable function, set its gradient equal to zero.

put $\nabla T = 0$ and solve for T, we will get ~~extr~~ mini or maxi value (extrema)

① Find the gradient of $r = \sqrt{x^2+y^2+z^2}$.

magnitude of the position vector.

$$\nabla r = \frac{\partial r}{\partial x}\hat{i} + \frac{\partial r}{\partial y}\hat{j} + \frac{\partial r}{\partial z}\hat{k}$$

grad

$$= \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right)\hat{i} + \frac{\partial}{\partial y}\left(\sqrt{x^2+y^2+z^2}\right)\hat{j} + \frac{\partial}{\partial z}\left(\sqrt{x^2+y^2+z^2}\right)\hat{k}$$

$$= \frac{2x}{2\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{2y}{2\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{2z}{2\sqrt{x^2+y^2+z^2}}\hat{k}$$

$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}}{r} = \hat{r}$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

class work.

Find the Gradient of

a) $f(x,y,z) = x^2+y^3+z^4$ $\nabla(x^2+y^3+z^4)$

b) $f(x,y,z) = x^2y^3z^4$

c) $f(x,y,z) = e^x \sin y \ln z$.

Assignment.

The height of a certain hill in feet is given by $h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$ where y is the distance (in miles) north, x the distance east of South Hadley.

a) Where is the top of the hill located? b) How high is the hill?

c) How steep is the slope at a point 1 mile north and one mile east of South hadley? In what direction is the slope steepest, at that point?

1.2.3 Operator ∇

Gradient '∇' has an apperance of a vector, Such that it 'multiplies' a scalar T

$$\nabla T = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) T$$

$$\nabla = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

But actually ∇ is not a vector. But when it 'acts' on a scalar, the answer will be a vector.

when an ordinary vector A ~~must~~ can multiply in 3 ways.

1. Multiply a scalar a: Aa
2. Multiply another vector B, via the dot product: A·B
3. Multiply another vector via the cross product: A×B.

There are 3 ways the ∇ can act:

1. On a scalar function T: ∇T (the gradient).
2. On a vector function V, via the dot product: $\nabla \cdot V$ (the divergence)
3. On a vector function V, via the cross product: $\nabla \times V$ (the curl)

1.2.4 The Divergence

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

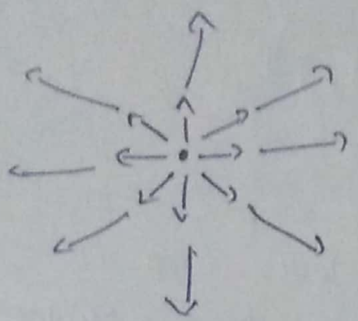
$$\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$$

$$\therefore \nabla\cdot\vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(V_x\hat{i} + V_y\hat{j} + V_z\hat{k}\right)$$

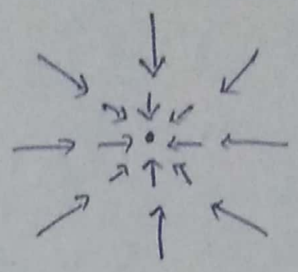
$$\nabla\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}.$$

Be careful, we can take divergence only with a vector (eg: $\vec{V}, \vec{A}, \vec{B}$). But answer will be a scalar value.

Geometrical Interpretation: The name divergence is well chosen, for $\nabla \cdot \vec{v}$ is a measure of how much the vector $\vec{v}$ spreads out (diverges) from the point in question.



arrow going out from a point
+ve divergence



· arrow going into the point
-ve divergence ·

Problem. (Do this during class time) and submit.

Let $V_a = r = x\hat{i} + y\hat{j} + z\hat{k}$.

$V_b = \hat{k}$

$V_c = z\hat{k}$

1) Find $\nabla \cdot V_a$, $\nabla \cdot V_b$, $\nabla \cdot V_c$.

2) Calculate the divergence of the following vector function.

a) $V_a = x^2\hat{i} + 3xz^2\hat{j} - 2xz\hat{k}$

b) $V_b = xy\hat{i} + 2yz\hat{j} + 3zx\hat{k}$

c) $V_c = y^2\hat{i} + (2xy + z^2)\hat{j} + 2yz\hat{k}$

1.2.5

· The curl.

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$$

$$\nabla \times \vec{v} = \hat{i}\left(\frac{\partial}{\partial y}V_z - \frac{\partial V_y}{\partial z}\right) - \hat{j}\left(\frac{\partial}{\partial x}V_z - \frac{\partial}{\partial z}V_x\right) + \hat{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

Note

Taking curl with only a vector and answer will also be a vector.

Physical Interpretation

$\nabla \times \vec{v}$, the name is curl.

$\nabla \times \vec{v}$ is a measure of how much the vector v "curls around" the point in question. Thus

? $V_a = -y\hat{i} + x\hat{j}$, $V_b = x\hat{j}$. Calculate the curl.

$$\nabla \times V_a = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = ?$$

$\nabla \times V_b = ?$

1.2.7

Second Derivatives

The gradient, the divergence, and the curl are the only first derivatives we can take with ∇

ie, ∇T, $\Rightarrow$ T scalar (gradient)

(divergence) $\nabla \cdot \vec{v} \Rightarrow$ v is a vector

(curl) $\nabla \times \vec{v}$

(Here $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$. $\Rightarrow$ first derivative)

By multiplying ∇ twice we can construct five types of second derivatives.

- The gradient ∇T is a vector, so we can take the divergence and curl of it

1. Divergence of gradient: $\nabla \cdot (\nabla T)$
2. Curl of gradient : $\nabla \times (\nabla T)$

we have $\nabla \cdot \vec{v} \rightarrow$ the answer is a scalar, so we can take its gradient.

3. $\nabla(\nabla \cdot \vec{v})$

The curl $\nabla \times \vec{v}$ is a vector, so we can take its divergence and curl.

$\nabla . \vec{v}$

(4) Divergence of curl: $\nabla \cdot (\nabla \times v)$

(5) curl of curl ; $\nabla \times (\nabla \times v)$

Take ①

$$\nabla \cdot \nabla T = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}\right)$$

$$= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$ is called as

Laplacian of T

Here T is a scalar

Now Laplacian of a vector $\vec{v}$

$$\nabla^2 \vec{v} = \nabla^2 v_x \hat{i} + \nabla^2 v_y \hat{j} + \nabla^2 v_z \hat{k}$$

② $\nabla \times \nabla T = 0$
curl of a gradient is always zero.

③ Actually $(\nabla \cdot \nabla) v = \nabla^2 v$

But $\nabla(\nabla \cdot v) \neq \nabla^2 v$

④ $\nabla \cdot (\nabla \times \vec{v}) = 0$

A x B x C

⑤ $\nabla \times (\nabla \times v) = \nabla(\nabla \cdot v) - \nabla^2 v$

Calculate the Laplacian.

a) $T_a = x^2 + 2xy + 3z + 4$

i.e. $\nabla^2 T_a = \dfrac{\partial^2 T_a}{\partial x^2} + \dfrac{\partial^2 T_a}{\partial y^2} + \dfrac{\partial^2 T_a}{\partial z^2}$

$= \dfrac{\partial^2}{\partial x^2}(x^2 + 2xy + 3z + 4) + \dfrac{\partial^2}{\partial y^2}(x^2 + 2xy + 3z + 4) +$

$\dfrac{\partial^2}{\partial z^2}(x^2 + 2xy + 3z + 4)$

First take $\dfrac{\partial}{\partial x}(\ )$, then again $\dfrac{\partial}{\partial x}\left[\dfrac{\partial}{\partial x}(\ )\right]$

$\therefore \nabla^2 T_a = 2$

b) $T_b = \sin x \sin y \sin z$

c) $T_c = e^{-5x} \sin 4y \cos 3z$

d) $V = x^2 \hat{i} + 3xz^2 \hat{j} - 2xz \hat{k}$