/6/2020 Chapter - 1

(1)

Electrodynamics

Study of electricity and magnetism was begun from 18^{th} century. i) Coulomb's observation on the forces b/w charges

then, ii) Gauss's law in Electrostatics.

Coulomb's law

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \hat{r}$$

Gauss's law in Electrostatics

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} \quad \text{or} \quad \nabla \cdot E = \rho/\varepsilon_0$$

Then we have studied that time varying current results in magnetic field. (observed by Oersted). And this relation b/w magnetic field and current was established by Biot and Ampere.

1. Biot - Savart's law

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{r}\, dl}{r^2}$$

2. Ampere's Law

$$\oint B \cdot dl = \mu_0 I_{enc} \quad \text{or} \quad \nabla \times \vec{B} = \mu_0 J$$

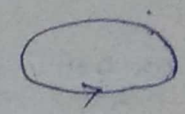
(2)

Faraday's Law:

Faraday discovered - Time varying mag. field results in dynamic ~~field~~ electric field (not static, as we have seen in electrostatics)

According to Faraday's law of e.m induction, if we consider any closed stationary path in space, ~~then~~ ~~which~~ an emf is induced in the path (a conducting wire loop), then this emf will be equal to -ve time rate of change of magnetic flux through the closed path.

$$\epsilon = -\frac{d\phi}{dt} \longrightarrow (1)$$



Here we can see that changing magnetic field produces an dynamic electric field that makes the charges to flow through the path.

e.m.f $\longrightarrow$ is the work done by the dynamic e field to move unit +ve charge once along the closed path.

(NB: e field present here is not same as $\nabla \times E = 0$ in electrostatics)

$$\epsilon = \oint_l E \cdot dl \longrightarrow (2)$$

workdone

comparing (1) and (2):

$$\oint_l E \cdot dl = -\frac{d\phi}{dt} \longrightarrow (3)$$

3

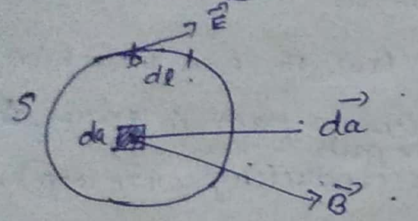
we have already studied,

$\phi \rightarrow$ mag. flux through closed path,

$$\phi = \int_S \vec{B}\cdot d\vec{a} \longrightarrow (4)$$

$\vec{B} \rightarrow$ mag. field intensity

$d\vec{a} \rightarrow$ small elemental area of a large surface S



Now substitute the value of ϕ in $\oint E\cdot dl = \frac{-d\phi}{dt}$.

$\therefore$ we get $\oint E\cdot dl = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{a} \longrightarrow (5)$

Suppose our surface is fixed (stationary) we can take the time derivate inside the integral.

$$\therefore \boxed{\oint_l E\cdot dl = -\int_S \frac{\partial B}{\partial t}\cdot d\vec{a}} \longrightarrow (6)$$

This is integral form of faraday's law.

just apply Stoke's theorm on LHS

the $\int_S (\nabla\times E)\cdot d\vec{a} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$

$$\boxed{\nabla\times\vec{E} = \frac{-\partial\vec{B}}{\partial t}} \longrightarrow (7)$$

Differential form of faraday's law.

4

This eq. no: (7) shows that $\bar{e}$ field produced here is not a conservative field $\rightarrow$ (becz curl of E does not vanishes) $\boxed{\text{if an } \bar{e} \text{ force field is conservative then } \nabla\times\vec{E} = 0}$.

here $\nabla\times E = -\frac{\partial B}{\partial t}$.

Now let's combine all the basic Laws in electricity & magnetics.

1) $\nabla\cdot E = \rho/\varepsilon_0 \longrightarrow$ Gauss's law in electrostatics

2) $\nabla\cdot\vec{B} = 0 \longrightarrow$ Gauss's law in magnetostatics

3) $\nabla\times\vec{E} = -\partial B/\partial t \longrightarrow$ Faraday's law

4) $\nabla\times\vec{B} = \mu_0 J \longrightarrow$ Ampere's law.

There are differential forms, corresponding integral forms are

1) $\int_S E\cdot da = Q/\varepsilon_0$

2) $\int B\cdot da = 0$.

3) $\int_l E\cdot dl = -\int \frac{\partial B}{\partial t}\cdot da$.

4) $\oint_l B\cdot dl = \mu_0 I_{enc}$!

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(5)

Induced electric field $(\vec{E})$ and Vector potential $(\vec{A})$.

Look at these equations

ie, $\nabla \times E = -\frac{\partial \vec{B}}{\partial t}$.

Induced $\vec{e}$ field depends on $-\frac{\partial B}{\partial t}$.

and from the eq. $\nabla \times B = \mu_0 J$.

mag. field depends on $\mu_0 J$.

So our Biot - Savarts Law, (mg. field)

$$\vec{B} = \frac{\mu_0}{4\pi}\int_V \frac{I \times \hat{r}}{r^2} d\tau \; ; \rightarrow (1) \text{ can be transformed to}$$

get $\vec{e}$ field.

$$\vec{E} = \frac{1}{4\pi}\int \frac{-\frac{\partial B}{\partial t} \times \hat{r}}{r^2} d\tau$$

$$\vec{E} = \frac{\partial}{\partial t}\left[-\frac{1}{4\pi}\int \frac{B \times \hat{r}}{r^2} d\tau\right] \longrightarrow (2)$$

Now, have,

$\nabla . B = 0$ and $\nabla \cdot A = 0$ mag - vector potential.

$\nabla \times \vec{B} = \mu_0 \vec{J}$ $\quad$ $\nabla \times \vec{A} = \vec{B}$

look. $\vec{A} \longrightarrow \vec{B}$ like $\vec{B} \longrightarrow \mu_0 \vec{J}$

So $\vec{B} = \frac{\mu_0}{4\pi}\int_V \frac{I \times \hat{r}}{r^2} d\tau$, similarly $\vec{A} = \int_V \frac{1}{4\pi} \frac{B \times \hat{r}}{r^2} d\tau$

(6)

$$A = \frac{1}{4\pi}\int \frac{B \times \hat{r}}{r^2} d\tau \longrightarrow (3)$$

Now look at eq. (2), substitute (3) in (2)

$$E = -\frac{\partial A}{\partial t} \longrightarrow (4)$$

$$\nabla \times E = -\nabla \times \frac{\partial A}{\partial t} \longrightarrow \nabla \times \vec{E} + \nabla \times \frac{\partial A}{\partial t} = 0$$

or $\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$ $\quad$ $\nabla \times (\quad) = 0$

(here $\nabla \times$ a vector $= 0$ (vanishes)

so this vector can be represented as -ve gradient of a scalar.)

Let take the scalar as V

$$\therefore E + \frac{\partial A}{\partial t} = -\nabla V$$

$$\boxed{\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}}$$

This eq. contains both static and dynamic components. Or this gives relation b/w induced electric field and vector potential.

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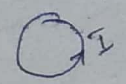
(7)

Inductance



Consider a steady state current I, flowing in a closed loop.

We have according to Bio-Savart's law mag. field set up is proportional to the current.

$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint_l \frac{\vec{dl} \times \hat{r}}{r^2}$$

If ~~$\vec{B}$~~ I is changed, $\vec{B}$ is also changed. change in $\vec{B}$ will make changes in magnetic flux ϕ associated with our loop, which is ($\phi = \vec{B} \cdot \vec{da}$)

~~We have already studied that according to faraday's law, the~~ turn induces an emf in it, according to faraday's law.

This induced emf has such a direction that ~~tends~~ tends to oppose the original change in current.

i.e. if the current in the circuit begins to decrease, it induces an e.m.f that opposes the decrease in current in the circuit. (or decreases the change in mag. flux).

(8)

So Inductance is the property that opposes any change in current in a closed circuit.

Lenz's law

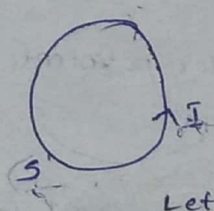
This law is the generalisation of the condition which we have just discussed.

This law i.e. Lenz's law state induced e.m.f will cause a current to flow in the closed loop in such a direction that opposes the change in the linking magnetic flux or change in current in the loop.

$$\epsilon = -\frac{d\phi}{dt}$$

Self Inductance (L)

Consider a closed circuit (loop) through which a current I flows. Then mag. flux linked with the circuit is



$$\phi = \int \vec{B} \cdot \vec{da} \longrightarrow (1)$$

We already know from Biot-Savart law that $\vec{B} \propto$ to current flowing. $B \propto I$, $\phi \propto B$

Let 's' is the surface of closed path. 's' is fixed surface

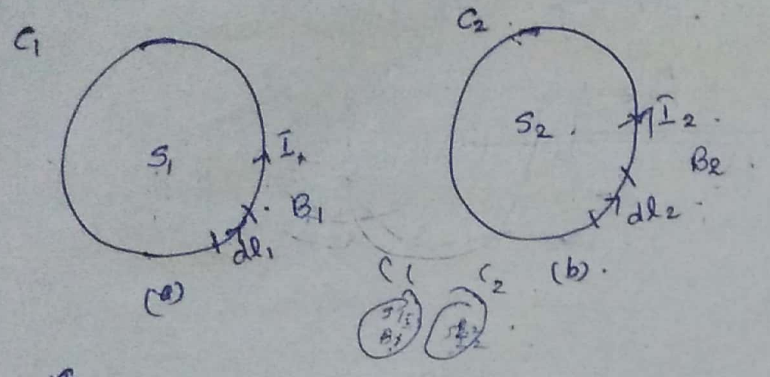
$B \propto I$
$B \propto \phi$
$\phi \propto I$

$\therefore \phi \propto I \longrightarrow (2)$

$\phi = LI \qquad L \longrightarrow$ self inductance.

$$\boxed{L = \phi / I}$$

Mutual inductance and Neumann formula.



Consider two closed loops C_1 and C_2 (they are very close to each other)

$I_1 \rightarrow$ current through C_1
$I_2 \rightarrow$ current through C_2.

this I_1 produces a mag. field of B_1.

$$B_1 = \frac{\mu_0 I_1}{4\pi} \oint_l \frac{dl_1 \times \hat{r}}{r^2} \quad (\longrightarrow \text{produced in } C_1)$$

Due to this B_1, the mag. flux 'ϕ' linked with C_2 will be

$$\boxed{\phi_{2\leftarrow 1} = \int_{S_2} \vec{B_1} \cdot \vec{da_2}} \rightarrow (1) \quad S_2 \rightarrow \text{surface of } C_2.$$

$B_1 \propto I_1$

$\therefore \phi_{2\leftarrow 1} \propto B_1 \propto I_1 \Longrightarrow \phi_{2\leftarrow 1} \propto I_1$

$\cancel{\phi_{2\leftarrow 1} \propto}$ or $\boxed{\phi_{2\leftarrow 1} = M_{21} I_1}$

This proportionality constant M_{21} is called mutual inductance of the two loops.

Similarly.

I_2 current in C_2 will produce B_2 mag. field.

$$B_2 = \frac{\mu_0 I_2}{4\pi} \oint_l \frac{dl \times \hat{r}}{r^2}.$$

Due to this B_2 flux ϕ' linked with C_1 will be

$$\boxed{\phi_{1\leftarrow 2} = \int_{S_1} B_2 \cdot \vec{da_1}} \longrightarrow (2) \qquad \phi = \int B \cdot da.$$

$B_2 \propto I_2$.

$\phi_{1\leftarrow 2} \propto B_2 \propto I_2$ $\qquad \therefore \phi_{1\leftarrow 2} \propto I_2$.

$$\boxed{\phi_{1\leftarrow 2} = M_{12} . I_2}$$

we have $\vec{B} = \nabla \times \vec{A}$.

$$\phi_{2\leftarrow 1} = \int_{S_2} (\nabla \times A_1) \cdot \vec{da_2}. \qquad \phi_{1\leftarrow 2} = \int_{S_1} (\nabla \times A_2) \cdot \vec{da_1}$$

apply stokes theorem

$$\phi_{2\leftarrow 1} = \oint_{C_2} A_1 \cdot dl_2. \qquad \phi_{1\leftarrow 2} = \oint_{C_1} A_2 \cdot dl_1$$

$$A_1 = \frac{\mu_0}{4\pi} \oint_{C_1} \frac{I_1}{r} dl_1 \qquad A_2 = \frac{\mu_0}{4\pi} \oint_{C_2} \frac{I_2}{r} dl_2.$$

$$\therefore \phi_{2\leftarrow 1} = \oint_{C_2} \left(\frac{\mu_0}{4\pi} \oint_{C_1} \frac{I_1}{r} dl_1\right) \cdot dl_2 \qquad \phi_{1\leftarrow 2} = \oint_{C_1} \left(\frac{\mu_0}{4\pi} \oint_{C_2} \frac{I_2}{r} dl_2\right) \cdot dl_1$$

$$\phi_{2\leftarrow 1} = \frac{\mu_0 I_1}{4\pi} \oint_{c_2} \oint_{c_1} \left(\frac{dl_1}{r}\right) \cdot dl_2 \longrightarrow (3)$$

$$\phi_{1\leftarrow 2} = \frac{\mu_0 I_2}{4\pi} \oint_{c_1} \left(\oint_{c_2} \frac{dl_2}{r}\right) \cdot dl_1 \longrightarrow (4)$$

we already have, eq.

$$\boxed{\phi_{2\leftarrow 1} = M_{21} I_1}$$

$$\boxed{\phi_{1\leftarrow 2} = M_{12} I_2}$$

compare this with (3)

$$\Rightarrow M_{21} = \frac{\mu_0}{4\pi} \oint_{c_2} \oint_{c_1} \frac{\vec{dl_1} \cdot \vec{dl_2}}{r}$$

$$M_{12} = \frac{\mu_0}{4\pi} \oint_{c_1} \oint_{c_2} \frac{\vec{dl_2} \cdot \vec{dl_1}}{r}$$

$$\boxed{M_{21} = M_{12} = M}$$

This eq. is known as Neumann's formula.

$M_{21} \rightarrow$ mutual inductance.

This mutual inductance depends only on size, shape and relative position of the two loops.

Also integral remains unchanged when c_1 and c_2 are interchanged.

$\therefore M_{21} = M_{12} = M$.

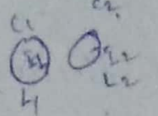
So, if through two near by loops (c_1 & c_2) same current is flowing, then mag. flux through loop, = mag. flux through loop 2.

i.e. $\phi_{1\leftarrow 2} = M_{12} I_2$; $\phi_{2\leftarrow 1} = M_{21} I_1$

$I_1 = I_2$, we already have $M_{12} = M_{21}$

$$\therefore \phi_{1\leftarrow 2} = \phi_{2\leftarrow 1}$$

Co-efficient of coupling.



Consider 2 loops.

I_1 current flows through loop 1 with L_1 self inductance.

I_2 current flows through loop 2 with L_2 self inductance.

and M is the mutual inductance between them.

we already have the equation for self inductance $L = \phi/I$

$\therefore L_1 = \dfrac{\phi_1}{I_1} \qquad L_2 = \dfrac{\phi_2}{I_2}.$ $\qquad \phi_1 = ? \Rightarrow$ mag. flux through loop 1

$\phi_2 = ? \Rightarrow$ " loop 2

Now, $\boxed{L_1 L_2 = \dfrac{\phi_1}{I_1}\dfrac{\phi_2}{I_2}} \longrightarrow (1)$

Also $M = M_{21} = M_{12}$.

also $\phi_{2\leftarrow 1} = M_{21} I_1 \quad \Rightarrow \boxed{M_{21} = \dfrac{\phi_{2\leftarrow 1}}{I_1}}$

$\phi_{1\leftarrow 2} = M_{12} I_2. \quad \boxed{M_{12} = \dfrac{\phi_{1\leftarrow 2}}{I_2}}$

Suppose the whole magnetic flux due to loop 1 is linked to loop 2,

then $\phi_{2\leftarrow 1} = \phi_1 \quad \longrightarrow M_{21} = \dfrac{\phi_1}{I_1}$ $\quad$ i.e, $\boxed{M = \dfrac{\phi_1}{I_1} = \dfrac{\phi_2}{I_2}}$.

|||y $\phi_{1\leftarrow 2} = \phi_2. \quad M_{12} = \dfrac{\phi_2}{I_2}$

Now look at eq. (1) $L_1 L_2 = MM \Rightarrow L_1 L_2 = M^2$.

or $\underline{M = \sqrt{L_1 L_2}}$.

But in actual practice whole ~~to~~ flux due to one loop cannot be linked to other. Some flux may lost b/w inter-linking.

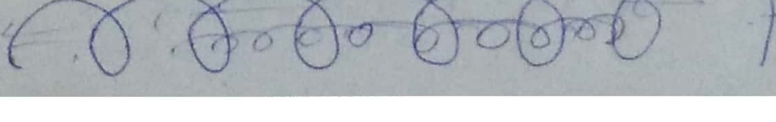
$\therefore M = k\sqrt{L_1 L_2}$.

$k \longrightarrow$ co-efficient of coupling ; value lies b/w 0 and 1.

when no leakage, its maximum value is 1.

Unit of inductance is Henries

Q. A short solenoid (length l, radius R and N_1 turns per unit length) lies on the axis of a very long solenoid (N_2 turns per length) What is the flux through the long solenoid and mutual inductance between the two solenoids.



Energy stored in a magnetic field :

a) In terms of current flowing.

we have, as $I \uparrow$, mag. field 'B' associated with the circuit also $\uparrow$. Faraday's law says that, this time varying mag. field induces an electric field (back e.m.f) which exerts force on $\bar{e}$ charges that constitute the current flow.

So to maintain the current flow work is to be done by the external voltage source against induced $\bar{E}$ field. This work is stored as potential energy, in mag. field.
stored in

Now, work done in moving a +ve charge along closed circuit once, $= -\mathcal{E}$

where $\mathcal{E}$ is the induced e.m.f. -ve sign shows that work is done against the emf.

If I is the current flowing, then total work done per unit time: $\frac{dw}{dt} = -\mathcal{E} I$; $\mathcal{E} = \frac{-d\phi}{dt}$; $\phi = LI$

$\therefore \mathcal{E} = -L\frac{dI}{dt} = \frac{-d(LI)}{dt}$

$$\therefore \frac{dw}{dt} = L\frac{dI}{dt} \cdot I \quad \text{or} \quad \boxed{\frac{dw}{dt} = IL\frac{dI}{dt}} \text{ ---- I}$$

If current is build up from 0 to I, work done,

$$w = \int_0^I dw = \int_0^I LI\,dI = L\int_0^I I\,dI = L\frac{I^2}{2}$$

$$\boxed{w = \frac{LI^2}{2}}$$

This work done is stored as energy in mag. field surrounding the closed conducting loop

b) In terms of mag. field. (15)

Mag. flux through a loop is given by

$$\phi = \int_S \vec{B}\cdot d\vec{a}$$

$$\phi = \int_S (\nabla\times\vec{A})\cdot d\vec{a} \qquad [B = \nabla\times\vec{A}]$$

C

Apply Stokes theorem

$$\phi = \oint_C A\cdot dl \longrightarrow (1)$$

Also, $\phi = LI \longrightarrow (2)$

Compare both; $LI = \oint_C \vec{A}\cdot d\vec{l} \longrightarrow (3)$

Substitute this in the eq. $\boxed{\omega = \frac{1}{2}LI^2}$; $\omega = \frac{1}{2}I(LI)$

$$\therefore \omega = \frac{1}{2} I \oint_C A\cdot dl.$$

I

dl

$$\boxed{\omega = \frac{1}{2}\oint_C (A\cdot I)\,dl}$$ (I and dl are the same direction)

We have studied that I is linear current flowing through line distribution.

Let us expand this for volume distribution of current with volume charge density J

$$\omega = \frac{1}{2}\int_V (A\cdot J)\,d\tau \rightarrow (1) \; ; \; \nabla\times B = \mu_0 J$$

$$\therefore J = \frac{1}{\mu_0}(\nabla\times B)$$

$$\therefore \omega = \frac{1}{2\mu_0}\int_V \vec{A}\cdot(\nabla\times B)\,d\tau \longrightarrow (2)$$

what is $(A\cdot\nabla\times B)$? for that we have,

$$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot\underbrace{(\nabla\times\vec{A})}_{B} - \vec{A}\cdot(\nabla\times\vec{B}).$$

$$\therefore A\cdot(\nabla\times\vec{B}) = \vec{B}\cdot\vec{B} - \nabla\cdot(\vec{A}\times\vec{B})$$

$$\vec{A}\cdot(\nabla\times\vec{B}) = B^2 - \nabla\cdot(\vec{A}\times\vec{B})$$ substitute in (2)

$$\omega = \frac{1}{2\mu_0}\int_V (B^2 - \nabla\cdot(\vec{A}\times\vec{B}))\,d\tau$$

$$\omega = \frac{1}{2\mu_0}\int_V B^2 d\tau - \frac{1}{2\mu_0}\int_V \nabla\cdot(\vec{A}\times\vec{B})\,d\tau$$

$$\frac{1}{2}\int_V (A\cdot J)\,d\tau = \frac{1}{2\mu_0}\int_V B^2 d\tau - \frac{1}{2\mu_0}\int_V \nabla\cdot(\vec{A}\times\vec{B})\,d\tau \longrightarrow (3)$$

change volume integral in II^{nd} term to surface integral using Gauss's divergence theorem.

$$\int_{v} \nabla\cdot(A\times B)\,d\tau \rightarrow \oint_{s}(A\times B)\cdot d\vec{a}$$

$$\boxed{\frac{1}{2}\int_{V}(A\cdot J)\,d\tau = \frac{1}{2\mu_0}\int_{V} B^2\,d\tau - \frac{1}{2\mu_0}\oint_{S}(\vec{A}\times\vec{B})\cdot d\vec{a}}$$

Look at this equation. I

LHS is constant, even if you increase the volume as you like, becz that volume includes the given current distribution.

Since LHS = RHS, RHS must also follow this.

But, as volume ↑, contribution from volume integral in RHS ↑ while contribution from surface integral ↓

So if we integrate ~~all~~ over all space, surface integral vanishes.

$$\therefore W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\,d\tau \implies \text{entire energy is stored in the magnetic field.}$$

Energy density in mag. field is either $\frac{B^2}{2\mu_0}$ or $\frac{1}{2}(A\cdot J)$

↓

$\frac{\text{energy}}{\text{volume}}$

Comparison:

Electric field	Mag. field
energy density $= \frac{1}{2}V\rho$	energy density $= \frac{1}{2}(A\cdot J)$
$= \frac{1}{2}\varepsilon_0 E^2$	$= \frac{B^2}{2\mu_0}$

Inconsistency of Ampere's law and Maxwell's modification

We already have the basic laws in electricity & magnetism.

i) $\nabla \cdot E = \rho/\varepsilon_0$

ii) $\nabla \cdot B = 0$

iii) $\nabla \times E = -\frac{\partial B}{\partial t}$

iv) $\nabla \times B = \mu_0 J$

Here we can see an inconsistancy in 4^{th} equation.

lets explain.

take eq-(iii) and find $\nabla \cdot \nabla \times E$

$$\nabla \cdot \nabla \times E = \nabla \cdot -\frac{\partial B}{\partial t}$$

$$\nabla \cdot \nabla \times E = -\frac{\partial}{\partial t} \nabla \cdot B$$

LHS is zero, since divergence of a curl is zero

and RHS is zero since $\nabla \cdot B = 0$.

$\therefore$ LHS = RHS

Now, take eq. iv and find $\nabla \cdot \nabla \times B$.

$$\nabla \cdot \nabla \times B = \nabla \cdot \mu_0 J$$

$$\nabla \cdot \nabla \times B = \mu_0 (\nabla \cdot J)$$

Here also LHS is zero, since divergence of curl is zero. we can write LHS = RHS

So RHS must be zero, then only

But, $\nabla \cdot J \neq 0$

ie, as per equation of continuity $\nabla \cdot J = -\frac{\partial \rho}{\partial t}$.

$\Longrightarrow$ RHS $\neq 0$

$\therefore$ eq. iv $\nabla \times B = \mu_0 J$ has an inconsistancy.

Let's see how maxwell modified it.

from Gauss's law $\nabla \cdot E = \rho/\varepsilon_0$

$$\rho = \varepsilon_0 (\nabla \cdot E)$$

Substitute in equation of continuity $\nabla \cdot J = -\frac{\partial}{\partial t} \rho$

$$\nabla \cdot J = -\frac{\partial}{\partial t} \varepsilon_0 (\nabla \cdot E)$$

$$\boxed{\nabla \cdot J = \nabla \cdot \varepsilon_0 \frac{-\partial E}{\partial t}}$$

So maxwell suggested that inconsistancy of eq. iv can be removed if $\varepsilon_0 \partial E/\partial t$ is added to original equation.

$\nabla \cdot \nabla \times E = \nabla \cdot -\frac{\partial B}{\partial t}$.

i.e, $\nabla \times B = \mu_0 \cdot (J + \varepsilon_0 \, \partial E/\partial t)$.

$\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \, \partial E/\partial t$.

Now, lets check for $\nabla \cdot \nabla \times B$

$\nabla \cdot \nabla \times B = \nabla \cdot (\mu_0 J + \mu_0 \varepsilon_0 \, \partial E/\partial t)$

$0 = \mu_0 [(\nabla \cdot J) + \nabla \cdot \varepsilon_0 \, \partial E/\partial t]$.

$\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \, \partial E/\partial t$

$\nabla \cdot \nabla \times B = 0$.

$\nabla \cdot \mu_0 J \neq 0$.

$\mu_0 (\nabla \cdot J) \neq 0$

$\nabla \cdot J = \partial \rho/\partial t$

we already have $\nabla \cdot J = \nabla \cdot \varepsilon_0 - \partial E/\partial t$.

$\therefore$ RHS $= \mu_0 (\nabla \cdot \varepsilon_0 - \partial E/\partial t + \nabla \cdot \varepsilon_0 \, \partial E/\partial t) = 0$.

Now inconsistancy is removed

$\therefore$ • Maxwells eauations for electromagnetism are

$$\nabla \cdot E = \rho/\varepsilon_0$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}.$$

Maxwell contributed an additional term $\varepsilon_0 \, \partial E/\partial t$.

This term $\varepsilon_0 \frac{\partial E}{\partial t} = \frac{\partial}{\partial t} \varepsilon_0 E = \frac{\partial \vec{D}}{\partial t}$.

and this $\frac{\partial D}{\partial t}$ is known as displacement current J_D.

$J_D = \frac{\partial D}{\partial t}$. displacement current density.

• Maxwells eq. in free space

In free space $\rho = 0$, $J = 0$

$\therefore$ eq. become

$\nabla \cdot E = \rho/\varepsilon_0$

$$\nabla \cdot E = 0$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t}.$$

$$\nabla \times B = \mu_0 \varepsilon_0 \, \partial E/\partial t$$

∇

Magnetic charge

Look at maxwells eq. in free space

$$\nabla \cdot E = 0 \longrightarrow (1)$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \longrightarrow (2)$$

$$\nabla \cdot B = 0 \longrightarrow (3)$$

$$\nabla \times B = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \longrightarrow (4)$$

There is some symmetry

eq. (3) and (4) are obtained by replacing E by B and B by $-\mu_0 \varepsilon_0 E$ in eq. (1) and (2).

But if we apply this symmetry in general Maxwell eq,

$$\nabla \cdot E = -\rho/\varepsilon_0$$

then $\nabla \cdot B \neq 0$; but a constant (must be)

$\frac{1}{\varepsilon_0}$ in electric field
μ_0 in magnetic field

$$\therefore \nabla \cdot B = \mu_0 \eta$$

and corresponding to $\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$

$$\nabla \times E = -\mu_0 K - \frac{\partial B}{\partial t}$$

where η is the magnetic charge density similar to electric charge density ρ

K is magnetic current density similar to electric current density J.

equation of continuity $\nabla \cdot K = -\frac{\partial \eta}{\partial t}$

(similar to $\nabla \cdot J = -\frac{\partial \rho}{\partial t}$)

So magnetic charge is a hypothetical entity obtained following the symmetry of maxwell's equations. Nobody can ever find ~~mag field~~ magnetic charge like electric charge.

Magnetic Vector potential and modified Ampere's law :

Ampere's law modified by Maxwell is given as

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

In terms of vector potential

$$\vec{B} = \nabla \times \vec{A}.$$

$$\nabla \times B = \nabla \times \nabla \times \vec{A} = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

$\nabla \times \vec{A}$

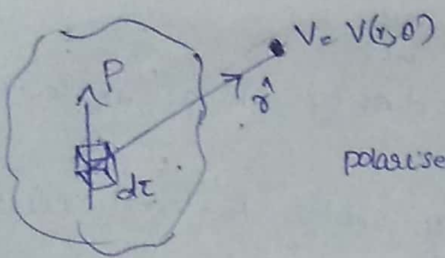
use BAC - CAB rule .

$$\boxed{\nabla(\nabla \cdot A) - \nabla^2 A = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}}$$

This is modified ampere's law in terms of magnetic vector potential

E.D part-5

The bound currents in a magnetized object .



The electric potential due to a polarised object is given by,

$$V = \frac{1}{4\pi\varepsilon_0} \int_{V'} \vec{P} \cdot \frac{\hat{r}}{r^2} \, d\tau$$

$\vec{P} \rightarrow$ polarization or induced ē dipole moment per unit volume of the object.

A similar relation can be obtained to find the magnetic potential due to a magnetised object.

ē potn But magnetic potential is a vector where as ē potential is a scalar. (look above eq.) So dot product in above eq. must be replaced by cross product.

If $\vec{A}$ is the magnetic vector potential and $\vec{M}$, the induced magnetic dipole moment per unit volume.

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{M} \times \hat{r}}{r^2} \, d\tau$$

$$\frac{\hat{r}}{r^2} = \nabla\left(\frac{1}{r}\right)$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_V \vec{M}\times\nabla.(\tfrac{1}{r})\,d\tau \longrightarrow (1)$$

$\frac{\mu_0}{4\pi}$ Let's change this $M\times\nabla(\frac{1}{r})$ into 2 terms.

using the identity $\nabla\times(f\vec{A}) = f(\nabla\times\vec{A}) - \vec{A}\times\nabla f$

$$\therefore A\times\nabla\vec{f} = f(\nabla\times\vec{A}) - \nabla\times f\vec{A}$$

compare with our eq.

$$\vec{M}\times\nabla(\tfrac{1}{r}) = \tfrac{1}{r}(\nabla\times\vec{M}) - \nabla\times\tfrac{1}{r}\vec{M} \longrightarrow (2)$$

Now substitute (2) in (1)

$$\vec{A} = \frac{\mu_0}{4\pi}\int_V\left(\frac{1}{r}(\nabla\times\vec{M}) - \nabla\times\frac{1}{r}\vec{M}\right)d\tau$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_V \frac{1}{r}(\nabla\times\vec{M})\,d\tau - \frac{\mu_0}{4\pi}\int_V\left(\nabla\times\frac{1}{r}\vec{M}\right)d\tau$$

Now second volume integral is changed to surface integral using $\int_V(\nabla\times F)d\tau = -\int_S F\times d\vec{a}$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_V \frac{1}{r}(\nabla\times\vec{M})\,d\tau - \frac{-\mu_0}{4\pi}\oint_S \frac{1}{r}(M\times d\vec{a})$$

look $d\vec{a} = da\,\hat{n}$.

$$\therefore \vec{A} = \frac{\mu_0}{4\pi}\int_V \frac{1}{r}(\nabla\times\vec{M})\,d\tau + \frac{\mu_0}{4\pi}\oint_S \frac{1}{r}(M\times\hat{n})\,da \longrightarrow (3)$$

Now look at these two terms, have you ever seen such a similar equation.

↳ Equation for potential of a volume charge ~~charge~~ density distribution in Electrostatics,

$$V_{vol,charge} = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho\,d\tau}{r}; \quad V = \frac{1}{4\pi\varepsilon}\, q/r$$

Compare this with first term in eq. (3)

Since Volume ~~char~~ current density in magnetostatics is analogous to volume charge density in electrostatics.

$$\therefore \vec{A}_{vol\,current} = \frac{\mu_0}{4\pi}\int \frac{J_b}{r}\,d\tau \quad \text{where } J_b = \nabla\times M.$$

Now look at second term, it look like the potential of a surface charge distribution.

$$V_{surf,charge} = \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\sigma\,da}{r}$$

Surface current density in magnetostatics is analogous to volume charge density in electrostatics:

$$\vec{A}_{surf\,current} = \frac{\mu_0}{4\pi}\int \frac{K_b}{r}\,da \qquad K_b = M\times\hat{n}$$

$$\therefore (3) \Rightarrow \vec{A} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J_b}\,d\tau}{r} + \frac{\mu_0}{4\pi}\oint_S \frac{\vec{K_b}}{r}\,da$$

$$J_b = \nabla \times M \qquad K_b = M \times \hat{n}$$

So the potential of a magnetised object will be the sum of potentials produced by the volume current density $J_b = \nabla \times M$ throughout the material and potential produced by the surface current $K_b = M \times \hat{n}$ over the surface.

This is actually similar to $\rho_b = -\nabla \cdot \vec{P}$

$\sigma_b = P \cdot \hat{n}$

in electrically polarised object.

ED-II part 6

Maxwell's equation inside matter.

In general, we have the maxwell's equations

$$\nabla \cdot E = \rho/\varepsilon_0$$

$$\nabla \times E = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}.$$

when we apply these equations to the materials which are subjected to electric and magnetic polarization, some modifications are needed. This is because, inside the polarised matter, there will be accumulation of bound charges and bound currents which cannot be directly controlled by an external agency.

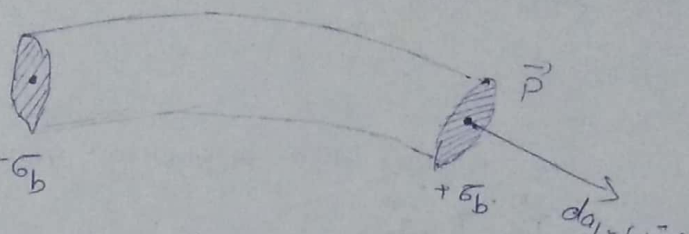
we have studied that due to electric polarisation $(\vec{P})$ bound charges accumulated in the matter. They are

$$\rho_b = -\nabla \cdot \vec{P}.$$

$|||^{ly}$ magnetic polarization or magnetisation $(\vec{M})$ results in a bound current

$$J_b = \nabla \times \vec{M}.$$

Consider a small piece of polarised material. The polarization induces a charge density ($+\sigma$ surface bound charge) at one end and $-\sigma_b$ at the other end.



If this polarization changes a little, the charge on each end also changes giving rise to a small current dI.

$$dI = \frac{\partial q}{\partial t} \qquad q = \sigma_b \, da_{\perp r} \quad \text{already studied} \quad \sigma_b = q/da_{\perp r}.$$

$$dI = \frac{\partial}{\partial t}(\sigma_b \, da_{\perp r}).$$

$$\frac{dI}{da_{\perp r}} = \frac{\partial \sigma_b}{\partial t}.$$

This $\frac{dI}{da_{\perp r}} = \vec{J}_p$, is the polarisation current density.

$$\therefore \quad |\vec{J}_p| = \frac{\partial \sigma_b}{\partial t} \longrightarrow (1)$$

we already have the eq. $\sigma_b = \vec{P}.\hat{n}$ ($\hat{n}$ → unit vector). Suppose P and $\hat{n}$ are parallel

$\vec{P}.\hat{n} = p\hat{n}\cos 0 = P\hat{n} = \vec{P}$ ($\hat{n}$ → unit vector = 1)

$\therefore \sigma_b = \vec{P} \qquad \therefore \partial\sigma_b = \partial\vec{P} \implies$ Change in end charges = change in polarisation

$\therefore \frac{\partial\sigma_b}{\partial t} = \left|\frac{\partial \vec{P}}{\partial t}\right| \longrightarrow$; eq. (1) $\implies |\vec{J}_p| = \left|\frac{\partial \vec{P}}{\partial t}\right| \longrightarrow (2)$

So it is to be noted that the polarization current (J_p) is the result of linear motion of charges due to change in electric polarisation while the bound current J_b is due to the magnetisation of the material involving spin and orbital motion of electrons.

Take divergence of eq. (2) $\quad \nabla \cdot J_p = \nabla \cdot \left|\frac{\partial \vec{P}}{\partial t}\right|$

$$\nabla \cdot J_p = \frac{\partial}{\partial t}(\nabla \cdot \vec{P}) \longrightarrow (3)$$

we already have the relation $\rho_b = -\nabla \cdot P$.

$$\therefore (3) \implies \boxed{\nabla \cdot J_b = -\frac{\partial \rho_b}{\partial t}} \longrightarrow 4$$

This equation $\implies$ equation of continuity for polarization current we have already studied that charge density in an electrically polarised object as the sum of free charge density ρ_f and bound charge density ρ_b.

$$\therefore \rho = \rho_f + \rho_b$$

$|||^y$ total current density in a magnetised material can be taken as

$$\vec{J} = \vec{J_f} + \vec{J_b} + \vec{J_p}$$

$J_f \rightarrow$ current density due to free charges

$J_b \rightarrow$ bound current density

$J_p \rightarrow$ polarisation current density

$$\therefore \left.\begin{array}{l} J_b = \nabla \times \vec{M} \\ J_p = \dfrac{\partial \vec{P}}{\partial t} \end{array}\right\} \text{already obtained}$$

$$\therefore J = J_f + (\nabla \times \vec{M}) + \frac{\partial \vec{P}}{\partial t} \longrightarrow (4)$$

substitute this in our 4^{th} Maxwell eq. $\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$

$$\therefore \nabla \times B = \mu_0 \left(J_f + (\nabla \times \vec{M}) + \frac{\partial \vec{P}}{\partial t} \right) + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

$$\nabla \times B = \mu_0 J_f + \mu_0 (\nabla \times \vec{M}) + \mu_0 \frac{\partial \vec{P}}{\partial t} + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

$$\nabla \times B - \mu_0 (\nabla \times \vec{M}) = \mu_0 J_f + \mu_0 \frac{\partial \vec{P}}{\partial t} + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \longrightarrow (5)$$

(5) $/ \mu_0$ $\qquad\qquad \vec{D} = \varepsilon_0 E$

$$\frac{\nabla \times B}{\mu_0} - \nabla \times \vec{M} = J_f + \frac{\partial \vec{P}}{\partial t} + \varepsilon_0 \frac{\partial E}{\partial t}$$

$$= J_f + \frac{\partial}{\partial t}(P + \varepsilon_0 E)$$

$$\nabla \left[\frac{B}{\mu_0} - \vec{M} \right] = J_f + \frac{\partial \vec{D}}{\partial t} \qquad \boxed{\vec{D} = P + \varepsilon_0 E}$$

$\downarrow$

electric displacement in a dielectric medium.

$$\nabla \times \left[B/\mu_0 - \vec{M} \right] = J_f + \frac{\partial \vec{D}}{\partial t} \longrightarrow (6)$$

if $\left[\frac{B}{\mu_0} - \vec{M} \right] = \vec{H}$, (6) $\Longrightarrow \nabla \times \vec{H} = J_f + \frac{\partial \vec{D}}{\partial t}$.

Here $\vec{H}$ is related to $\vec{B}$ as $\vec{D}$ is related $\vec{E}$.

$\vec{H}$ depends only on current, like $\vec{D}$ depends only on charge

But $\vec{B}$ depends on both current and magnetisation of medium

$\vec{E}$ " charge " polarization "

In non magnetic medium;

$$\vec{H} = \frac{\vec{B}}{\mu_0} \quad \text{i.e, } \vec{B} = \mu_0 \vec{H}$$

For magnetic medium

$$\vec{H} = \frac{\vec{B}}{\mu_0} - M$$

For linear magnetic medium

$$\vec{B} = \mu H$$

Other Maxwell's equations ($\nabla \times E$ and $\nabla \cdot \vec{B}$) do not require any modification becz, it is independent of ρ and J.

$$\nabla \cdot E = \rho/\varepsilon_0 \implies \nabla \cdot \underbrace{(\varepsilon_0 E)}_{D} = \rho$$

$$\nabla \cdot D = \rho_f$$

∴ The maxwells equations inside matter (polarised and magnetised material) are

$$\boxed{\begin{aligned} \nabla \cdot \vec{D} &= \rho_f \\ \nabla \times E &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \cdot B &= 0 \\ \nabla \times \vec{H} &= J_f + \frac{\partial \vec{D}}{\partial t} \end{aligned}}$$

ED part-7

Boundary Conditions

- We are using integral forms of Maxwells equations to see what happens at the boundary b/w 2 media.

The conditions are as follows

i. Tangential or parallel component of E is cts at the boundary.
ii) " of H "
(Not true if boundary is a perfect conducting surface)
iii) Normal component of B is cts at the boundary.
iv) Normal component of D is cts at the boundary if there is no surface charge.

Pb1

Integral form of Maxwell's equations in a medium

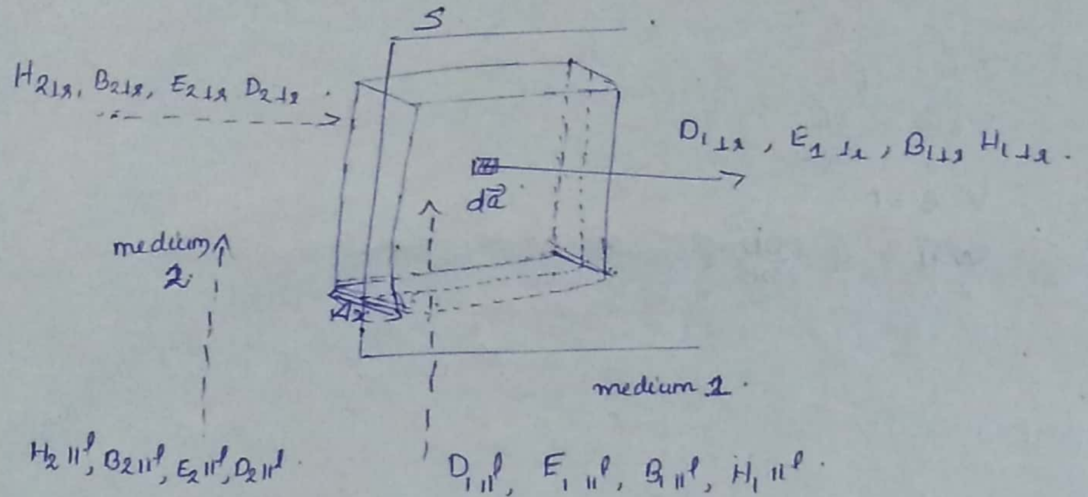
$\nabla\cdot D = \rho_f \implies \oint_S \vec{D}\cdot d\vec{a} = Q_f$. ⟶ (1) $\quad Q_f = \int \sigma_f \, da$

$\nabla\cdot B = 0 \implies \oint_S B\cdot d\vec{a} = 0$ ⟶ (2)

$\nabla\times\vec{E} = -\frac{\partial B}{\partial t} \implies \oint_l E\cdot dl = \frac{\partial}{\partial t}\oint_S \vec{B}\cdot d\vec{a}$ ⟶ (3)

$\nabla\times H = J_f + \frac{\partial D}{\partial t} \implies \oint H\cdot dl = I_f + \frac{\partial}{\partial t}\oint_S \vec{D}\cdot d\vec{a}$ ⟶ (4)

Let's consider a boundary 'S' abt which a wafer thin pill box extending on either side of the boundary is constructed



(1) Boundary condition for $\vec{D}$

$$\oint \vec{D}\cdot d\vec{a} = Q_f = \int_S \sigma_f \, da$$

Here only surface charge exists. Apply this to 3 parts

$$\oint_S D\cdot da = \int_{\text{flat surface (1)}} \vec{D}\cdot d\vec{a} + \int_{\text{flat surface (2)}} \vec{D}\cdot d\vec{a} + \int_{\text{edges}} \vec{D}\cdot d\vec{a} = \int_S \sigma_f \, da$$

Look at edges - Let area of pill box $=0$ i.e. $\Delta x = 0$

$$\therefore \int_{\text{edge}} D\cdot d\vec{a} = 0$$

$$\therefore \oint_S D\cdot da = \int_{\text{flat } S(1)} D\cdot da + \int_{\text{flat } (2)} D\cdot da = \int \sigma_f \, da$$

This can be written as

$$\oint_{f.S(1)} D_{1\perp r} \cdot da - \oint_{f.S(2)} D_{2\perp r} \cdot da = \int \sigma_f \, da$$

Look to the figure $D_{1\perp r}$ & da parallel
$D_{2\perp r}$ & da.

$$\therefore \int D_{1\perp r} \, da - \int D_{2\perp r} \, da = \int \sigma_f \, da$$

$$\therefore \boxed{D_{1\perp r} - D_{2\perp r} = \sigma_f} \Rightarrow$$ $\perp r$ component of D is discontinuous by an amount σ_f.

If $\sigma_f = 0$ then $D_{1\perp r} - D_{2\perp r} = 0 \Rightarrow D_{1\perp r} = D_{2\perp r}$.

ie, Normal component is cts only if boundary is without charge

② Boundary condition for $\vec{B}$

$$\oint_S B \cdot da = 0$$

$$\oint_S B \cdot da = \int_{f.S(1)} B \cdot da + \int_{f.S(2)} B \cdot da + \int_{edge} \vec{B} \cdot \vec{da} = 0$$

Area of pill box at edges $= 0$ ($\Delta x \to 0$)

$$\therefore \int_{edges} B \cdot da = 0$$

$$\therefore \oint_S B \cdot da = \int \vec{B}_{1\perp r} \cdot \vec{da} - \int B_{2\perp r} \cdot \vec{da} = 0$$

In flat surface $B_{1\perp r}$ and $\vec{da}$ are parallel.

$$\therefore \int B_{1\perp r} \, da - \int B_{2\perp r} \, da = 0$$

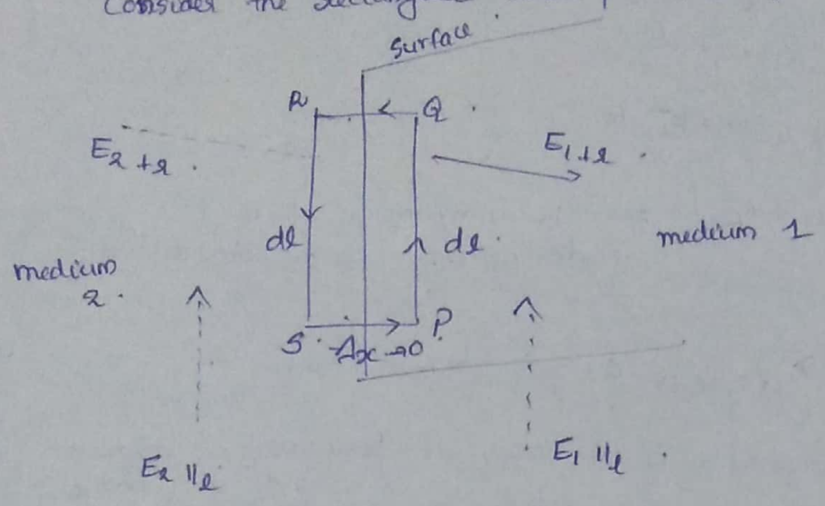
or $B_{1\perp r} - B_{2\perp r} = 0 \Rightarrow \boxed{B_{1\perp r} = B_{2\perp r}}$.

Normal component of B is cts across the boundary.

Boundary Condition for $\vec{E}$

$$\oint_l E \cdot dl = -\frac{\partial}{\partial t} \oint \vec{B} \cdot \vec{da}$$

Consider the rectangular closed path PQRSP



$$\oint_l E \cdot dl = \oint_{PQRSP} E \cdot dl$$

$$= \int_{PQ} E \cdot dl + \int_{QR} E \cdot dl + \int_{RS} E \cdot dl + \int_{SP} E \cdot dl$$

$$= \int_{PQ} E_{1\parallel} \cdot dl + \int_{QR} (E_{1\perp l} + E_{2\perp l}) \cdot dl + \int_{RS} E_{2\parallel l} \cdot dl + \int_{SP} (E_{2\perp} + E_{1\perp}) \cdot dl$$

$$= -\frac{\partial}{\partial t} \oint \vec{B} \cdot \vec{da}$$

QR and SP $\to 0$

$\therefore$ IInd and IVth term $\to 0$

also we have $\oint B \cdot da = 0$ Becz surface integral is taken over the surface enclosed by PQRSP for which area $\to 0$ $(SP \to 0,\ QR \to 0)$

$$\oint_{PQRSP} E \cdot dl = \int_{PQ} E_{1\parallel l} \cdot dl + \int_{RS} E_{2\parallel l} \cdot dl = 0$$

look direction

$$\therefore E_{1\parallel l} \int dl - E_{2\parallel l} \int dl = 0$$

$PQ = RS = l$

$$\therefore E_{1\parallel l}\, l - E_{2\parallel l}\, l = 0$$

$$\Rightarrow \boxed{E_{1\parallel} l = E_{2\parallel} l}$$

Parallel or tangential component of E is cts

Boundary condition for $\vec{H}$.

$$\oint H \cdot dl = I_f + \frac{\partial}{\partial t}\int_s \vec{D}\cdot\vec{da}$$

$$\int \vec{D}\cdot\vec{da} = 0$$

Using pill box edge area enclosed PQRSP.

$$\oint_{PQRSP} H \cdot dl = \int_{PQ}(H_{1_{\parallel}}\cdot dl) + \int_{QR}(H_{1_{\perp}} + H_{2_{\perp}})\cdot dl + \int_{RS} H_{2\parallel}\cdot dl + \int_{SP}(H_{2_{\perp}} + H_{1_{\perp}})\cdot dl = I_f$$

$$QR = SP = 0 \quad (\Delta x \rightarrow 0)$$

$$\therefore \oint_{H\cdot dl} = \oint_{PQRSP} H\cdot dl = \int_{PQ} H_{1\parallel}\cdot dl + \int_{RS} H_{2\parallel}\cdot dl$$

$$= H_{1\parallel}\int_{PQ} dl - H_{2\parallel}\int_{RS} dl = I_f$$

$PQ = RS = l$.

$$H_{1\parallel} l - H_{2\parallel} l = I_f.$$

I_f is the free current enclosed by the amperian loop. Here the volume current does not exist as no volume is enclosed by PQRSP. So I_f will be only due to free surface current.

Surface current density $k_f = \frac{dI_f}{dl_{\perp}}$

$$dI_f = k_f\, dl_{\perp} = k_f\cdot(\hat{n}\times \hat{dl})\, dl$$

$$dI_f = (k_f \times \hat{n})\cdot dl$$

$\hat{n}$ - unit vector $\perp$ to surface

k_f $\parallel$ to surface.

$k_f \times \hat{n}$ is parallel to sides PQ & RS of the amperian loop

$$\int_l (k_f \times \hat{n})\cdot dl = \int dI_f.$$

$$I_f = |k_f \times \hat{n}|\, l$$

$$\therefore H_{1\parallel} l\ H_{2\parallel} l = |k_f \times \hat{n}|\, l.$$

$$H_{1\parallel} - H_{2\parallel} = |k_f \times \hat{n}|$$

ie, Tangential component or $\parallel$ component of H is ~~the~~ discontinuous.

Summary

B.C

1) $D_{1\perp r} - D_{2\perp r} = \sigma_f$ or $\varepsilon_1 E_{1\perp r} - \varepsilon_2 E_{2\perp r} = \sigma_f$

2) $B_{1\perp r} - B_{2\perp r} = 0$

3) $E_{1\parallel l} - E_{2\parallel l} = 0$

4) $H_{1\parallel l} - H_{2\parallel l} = |k_f \times \hat{n}|$ or $\frac{B_{1\parallel l}}{\mu_1} - \frac{B_{2\parallel l}}{\mu_2} = |k_f \times \hat{n}|$

ED-II part 8

Potential formulation in Electrodynamics

1. Scalar and Vector Potentials

We have the basic Maxwell's equations are

$\nabla \cdot E = \rho/\varepsilon_0 \longrightarrow (1)$

$\nabla \cdot B = 0 \longrightarrow (2)$

$\nabla \times E = -\frac{\partial B}{\partial t} \longrightarrow (3)$

$\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \longrightarrow (4)$

These are first order differential equations connecting $\vec{E}$ and $\vec{B}$

eq. (3), ie, $\vec{E}$ electric field, ~~to~~ is not possible to describe in terms of only Scalar potential.

In terms of magnetic Vector Potential ($\vec{A}$)

$$\vec{B} = \nabla \times \vec{A}$$

$$\therefore \nabla \times E = -\frac{\partial B}{\partial t}$$

$$\Rightarrow \nabla \times E = -\frac{\partial}{\partial t} \nabla \times \vec{A}$$

$$\nabla \times E = -\nabla \times \frac{\partial A}{\partial t}$$

$$\nabla \times E + \nabla \times \frac{\partial A}{\partial t} = 0$$

$$\nabla\times\left(E+\frac{\partial A}{\partial t}\right)=0 \longrightarrow ①$$

∴ This is a vanishing curl

We have an equation $\nabla\times(-\nabla V)=0 \longrightarrow ②$

($-\nabla V \Rightarrow$ static electric field and also curl of grad of scalar function is zero) compare ① and ②

$$\therefore E+\frac{\partial A}{\partial t}=-\nabla V$$

$$\vec{E}=-\nabla V-\frac{\partial \vec{A}}{\partial t}.$$

$\Rightarrow$ An active electric field can be obtained by combining scalar and vector potentials.

Gauss's law and Ampere's law considering the consistency of Maxwell's equations with scalar and vector potentials

Gauss's law is.

$$\nabla\times B=\mu_0 J$$
$$\nabla\times B=\mu_0 J+\mu_0\varepsilon_0\,\partial E/\partial t$$

$$\nabla\cdot\vec{E}=\rho/\varepsilon_0$$

$$\vec{E}=-\nabla V-\frac{\partial \vec{A}}{\partial t}.$$

$$\nabla\cdot\left(-\nabla V-\frac{\partial \vec{A}}{\partial t}\right)=\rho/\varepsilon_0$$

$$\nabla\cdot\nabla V=-\rho/\varepsilon_0$$

$$+\nabla(\nabla V)+\nabla\cdot\frac{\partial \vec{A}}{\partial t}=-\rho/\varepsilon_0 \longrightarrow ①$$

eq. ① reduces to Poisson's equation in static case where current 0, $\vec{A}=0$

$$\therefore \nabla^2 V=-\rho/\varepsilon_0$$

Modified Ampere's equation is

$$\nabla\times\vec{B}=\mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}, \qquad B=\nabla\times\vec{A}, \quad E=-\nabla V-\frac{\partial\vec{A}}{\partial t}.$$

$$\nabla\times\nabla\times\vec{A}=\mu_0 J+\mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\nabla V-\frac{\partial A}{\partial t}\right)=$$

$$\nabla(\nabla\cdot A)-\nabla^2 A=\mu_0 J-\mu_0\varepsilon_0\nabla\frac{\partial V}{\partial t}-\mu_0\varepsilon_0\frac{\partial^2 A}{\partial t^2}.$$

$$-\mu_0 J=-\underline{\nabla(\nabla\cdot A)}+\nabla^2 A-\underline{\mu_0\varepsilon_0\nabla\,\partial V/\partial t}-\mu_0\varepsilon_0\,\partial^2 A/\partial t^2$$

$$-\mu_0 J = \nabla^2\vec{A} - \mu_0\varepsilon_0 \frac{\partial^2\vec{A}}{\partial t^2} - \nabla\left(\nabla\cdot A + \mu_0\varepsilon_0\frac{\partial V}{\partial t}\right) \longrightarrow (2)$$

Adding and substracting $\mu_0\varepsilon_0 \frac{\partial^2 V}{\partial t^2}$ to L.H.S of eq. (1)

ie, $\nabla\cdot(\nabla V) + \frac{\partial}{\partial t}(\nabla\cdot\vec{A}) = -\rho/\varepsilon_0$

$\nabla\cdot E = -\rho/\varepsilon_0$

$E = -\nabla V - \partial\vec{A}/\partial t$

$$\nabla^2 V - \mu_0\varepsilon_0\frac{\partial^2 V}{\partial t^2} + \frac{\partial}{\partial t}\left(\nabla\cdot\vec{A} + \mu_0\varepsilon_0\frac{\partial V}{\partial t}\right) = -\rho/\varepsilon_0 \longrightarrow (3)$$

So,

$$\nabla^2\vec{A} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \nabla\left(\nabla\cdot\vec{A} + \mu_0\varepsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0 J$$

$$\nabla^2 V - \mu_0\varepsilon_0\frac{\partial^2 V}{\partial t^2} + \frac{\partial}{\partial t}(\nabla\cdot\vec{A}) + \mu_0\varepsilon_0\frac{\partial V}{\partial t}) = -\rho/\varepsilon_0$$

These two are Maxwell's equations expressed in terms of E-D potentials V and $\vec{A}$.

These two equations contains all the informations available from Maxwell's equation

Assignment

Gauge Transformations