Exact Differential equations

Defn A differential equation is said to be exact if it can be derived from its primitive directly by differentiation without any subsequent multiplication, elimination etc.

Thus the differential equation

$M(x, y) + N(x, y)\, y' = 0$ is exact

Iff $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

and it can be integrated as follows.

Firstly integrate M w.r to x, regarding y as a constant.
Then integrate w.r to y those terms in N which do not involve x. The sum of two expression is the required solution.

eg :- Show that the equation

$$(1+4xy+2y^2)\,dx + (1+4xy+2x^2)\,dy = 0$$

is exact and solve it?

soln :- here $M = 1+4xy+2y^2$

$N = 1+4xy+2x^2$

$$\frac{\partial M}{\partial y} = 0+4x+4y.$$

$$\frac{\partial N}{\partial x} = 0+4y+4x.$$

i.e. $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 4x+4y,$$

∴ the equation is exact.

Soln is $\int M$ w.r to x + $\int$ terms in N which do not involve x w.r to y = c.

$$\int 1+4xy+2y^2\,dx + \int 1\,dy = c$$

$$x+2x^2y+2y^2x+y = c$$

Integrating factors:

Some times, a given D.E $P(x,y)dx + Q(x,y)dy = 0$ is not exact but can be made exact by multiplying it by a suitable function $u(x,y)$, not identically equal to zero. This function is called an integrating factor.

Remark :- Integrating factor of a DE need not be unique.

Method of finding integrating factors

Rule 1 :- If a DE $P(x,y)dx + Q(x,y)dy = 0$ is such that

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

is a function of x alone, say $f(x)$
then $e^{\int f(x)dx}$ is an integrating factor of the above D.E.

Rule 2 :- If the differential equation $P(x,y)dx + Q(x,y)dy = 0$ is such that $\frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$ is a function of y alone, say $f(y)$, then $e^{\int f(y)dy}$ is an integrating factor of the above D.E.

eg :- solve. $(x^2 - 2x + 2y^2)dx + 2xy\,dy = 0$

here $P = x^2 - 2x + 2y^2$ & $Q = 2xy$.

$\frac{\partial P}{\partial y} = 4y$ $\qquad$ $\frac{\partial Q}{\partial x} = 2y$.

$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, $\Rightarrow$ not exact.

Now evaluate

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$$

$$= \frac{1}{2xy}(4y-2y)$$

$$= \frac{1}{2xy} \times 2y$$

$= 1/x$, function of x alone.

hence Integrating Factor (I F)

$$= e^{\int f(x)dx} = e^{\int 1/x \, dx}$$

$$= e^{\ln x} = \underline{\underline{x}}.$$

multipling the given equn throughout by IF - x we get.

$$(x^3 - 2x^2 + 2xy^2)\,dx + 2x^2y\,dy = 0.$$

$\rightarrow$ which is exact (verify).

$\therefore$ soln is $\int P$ w.r to x + $\int$ terms in Q not inside x w.r to y = C

$$\frac{x^4}{4} - \frac{2x^3}{3} + \frac{2y^2x^2}{2} + 0 = c.$$

$$\frac{x^4}{4} - \frac{2x^3}{3} + x^2y^2 = c$$

2) show that the differential eqn $(2xy + y - \tan y) + (x^2 - x\tan^2 y + \sec^2 y)y' = 0$ is exact and solve it.

3. $(2x+3) + (2y-2)y' = 0$

4 $2x + 4y + (2x - 2y)y' = 0$

5 $(y - x^2)dx + (x^2\sin y - x)dy = 0$

6 $x\,dy - y\,dx = (1+y^2)dy$.

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2) $M = 2xy + y - \tan y$

$N = x^2 - x\tan^2 y + \sec^2 y + 2$

$$\frac{\partial M}{\partial y} = 2x + 1 - \sec^2 y.$$

$$\frac{\partial N}{\partial x} = 2x - \tan^2 y$$

$$= 2x + (1 - \sec^2 y)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{given eqn is exact.}$$

Soln is $\int M$ w r to x + $\int$ terms in N not involving x w.r to $y = c$

$2\frac{x^2}{2}y + yx - x\tan y + \tan y + 2y = c$

$x^2 y + xy - x\tan y + \tan y + 2y = c$

3) $(2x+3)\,dx + (2y-2)\,dy = 0$

$$\frac{\partial M}{\partial y} = 0 \qquad \frac{\partial N}{\partial x} = 0$$

$\Rightarrow$ exact.

soln $\int (2x+3)\,dx + \int (2y-2)\,dy = c$

$$2x^2/2 + 3x + 2y^2/2 - 2y = c$$

$$x^2 + 3x + y^2 - 2y = c$$

4) $(2x+4y)\,dx + (2x-2y)\,dy = 0$

$$\frac{\partial M}{\partial y} = 4 \qquad \frac{\partial N}{\partial x} = 2.$$

$1/x-y$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

$\Rightarrow$ not exact.

$\frac{1}{P}\left(\partial_{M} \ldots\right)$

$e^{\int f \ldots}$

Linear and Nonlinear differential Equation

The Ordinary d.e of order n $F(t, y, y', \ldots y^n) = 0$ is said to be linear if F is a linear function of the variables $y, y' \ldots y^n$.

examples

① $$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

② $$t^2\frac{\partial^2 u}{\partial t^2} - x\frac{\partial y}{\partial x} - \sin t\,\frac{\partial u}{\partial t} = u.$$

3. $$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y^2 = 0$$

4) $$\frac{d^2y}{dx^2} + 5y\frac{dy}{dx} + 6y = 0.$$

here ③ is nonlinear because the dependent variable y appears to the second degree i.e. y^2.

In equation (4) the term $5y\frac{dy}{dx}$, which involve the product of the dependent variable and its derivative. hence it is non linear.

defn A differential equation is said to be linear when the dependent variable and its derivatives occur only in the first degree and are not multiplied together.

The most general form of a linear equation of the first order is

$y' + p(x)y = g(x)$.

If $g(x)$ is zero $\forall x$ in the given interval, the equation is said to be homogeneous, otherwise it is non homogeneous.

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Solution of homogeneous linear differential Equation

The homogeneous linear equation of first order $y' + p(x)y = 0$ —— (1) can be written as $\frac{dy}{dx} = -p(x)y$

$$\frac{dy}{y} = -p(x)\,dx.$$

~~on~~ Integrating both sides.

$$\ln|y| = -\int p(x)\,dx + k.$$

taking exponentials we g.

$$y = ce^{-\int p(x)\,dx}.$$

This is soln of homogeneous equn (1).

solutions of non homogeneous differential equation

The general solution of the non homogeneous linear equation is

$$y \cdot e^{\int p(x)dx} = \int g(x) e^{\int p(x)dx} + c.$$

eg:- solve $y' - 2y = 4 - t$ $\quad \boxed{y' + p(x)y = g(x)}$

here $g(t) = 4 - t$

$p(t) = -2$.

$e^{\int p dt} = e^{\int -2 dt} \longrightarrow I.F.$

$= e^{-2t}$

soln, $y e^{\int p dt} = \int g(t) e^{\int p dt}$

$y \cdot e^{-2t} = \int (4-t) e^{-2t}$

$y e^{-2t} = \frac{4e^{-2t}}{-2} - \frac{te^{-2t}}{-2} + \frac{e^{-2t}}{4} + c$

$y e^{-2t} = -2e^{-2t} + \frac{te^{-2t}}{2} + \frac{1}{4} e^{-2t} + c$

(2) $y' + y\tan x = \cos^3 x$.

$P(x) = \tan x$

$$e^{\int P(x)dx} = e^{\int \tan x\, dx}$$

$$= e^{\ln|\sec x|}$$

$$= \sec x.$$

$$y \cdot e^{\int P(x)dx} = \int \cos^3 x \cdot \sec x\, dx$$

$$y \cdot \sec x = \int \cos^2 x\, dx.$$

$$= \int \frac{1 + \cos 2x}{2}\, dx$$

$$= \frac{1}{2}x + \frac{\sin 2x}{4}.$$

$$y \sec x = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

Method of variation of Parameter

This is the method, for finding the general solution of first order linear d.e. $y' + p(t)y = g(t)$. —(1)

First consider the homogeneous d.e $y' + p(t)y = 0$.

we know that the general solu is

$$y = ce^{-\int p(t)\,dt},$$

$\therefore$ the general solution of (1) is of the form $y = u(t)e^{-\int p(t)\,dt}$.

where u is a function of t.

substituting y and y' in (1) and solve we get the general solution eg. solve $y' - (2/t)\, y = t^2 \cos 3t$ $\{ y' + p(t) y = \ldots$

corresponding homogeneous d.e

is $$y' - (2/t)\, y = 0 \quad \text{—} \; (2)$$

here $p(t) = -2/t$

$$\therefore \; y = c e^{-\int p(t)\,dt}$$

$$= c e^{-\int -2/t \; dt}$$

$$= c e^{+2 \ln t}$$

$$= c e^{\ln t^2}$$

$$= \underline{\underline{c t^2}} \quad \text{—} \; (3).$$

hence the general solution of

(2) is

$$y = c e^{-\int p(t)\,dt} = c t^2 \, , \quad c \text{ is a constant}$$

Let the general soln of ① is
$y = u t^2$, where u is a function of t.

substituting for y from ③ in ① we obtain.

$y' = u \times 2t + u' t^2$;

∴ ① becomes
$2u(t)t + u't^2 - (2/t)u(t)t^2 = t^2\cos 3t$

$2u(t)t + u't^2 - 2u(t)t = t^2\cos 3t$

$\Rightarrow u't^2 = t^2\cos 3t$

$\Rightarrow u' = \cos 3t$

$\Rightarrow u = \frac{\sin 3t}{3} + K$

hence the general soln of ① is

$y = u(t)t^2$

$y = \left[\frac{\sin 3t}{3} + K\right]t^2$

Q2 :- solve $y' - y = 3e^{t}$ — (1)

corresponding homogeneous de.

$$y' - y = 0$$

$$y = ce^{-\int p(t)dt}$$

$$= ce^{-\int -1\, dt} = ce^{t}$$

hence solution of (1) is

$$y = u(t)\, e^{t}$$

$$y' = ue^{t} + e^{t}u'$$

$\therefore$ (1) becomes

$$\cancel{ue^{t}} + e^{t}u' - \cancel{ue^{t}} = 3e^{t}$$

$$u'e^{t} = 3e^{t}$$

$$u' = 3$$

$$\Rightarrow u = 3t + k, \quad k \text{ is const}$$

hence general soln of (1) is

$$y = (3t + k)e^{t}$$

Bernoulli's Equation

A d.e of the form

$y' + p(x)y = g(x)y^a$ where $p(x)$ & $g(x)$ are function of x alone and a is any real number is called Bernoulli's equation

To solve Bernoulli's equation, dividing the above eqn by y^a we get

$$y^{-a}y' + p(x)y^{1-a} = g(x) \quad \text{--- (1)}$$

in (1) put $u = y^{1-a}$ Then

$$u' = (1-a)y^{-a}y'$$

$\therefore$ (1) becomes $\dfrac{u'}{1-a} + p(x)u = g(x)$.

$$u' + (1-a)p(x)u = g(x)(1-a)$$

This is a linear equation in u and hence solve this

1) solve $y^2 y' - y^3 \tan x = \sin x \cos^2 x$. — (1)

general form of Bernoulli's Equa is $y' + P(x) y = g(x) y^a$.

dividing (1) throughout by y^2

$$y' - y \tan x = \frac{\sin x \cos^2 x}{y^2}$$

$$y' - y \tan x = \sin x \cos^2 x \cdot y^{-2}.$$

This is a Bernoulli's equn with

$a = -2$.

dividing the above equn through out by y^{-2} we get

$$y^2 y' - \underline{y^3} \tan x = \sin x \cos^2 x$$ — (2)

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put $y^3 = u$.

Then $3y^2 y' = u'$

$\therefore$ (2) becomes.

$$\frac{u'}{3} - u \tan x = \sin x \cos^2 x.$$

$$u' - 3u \tan x = 3 \sin x \cos^2 x. \quad (3)$$

coefficient of $u = p(x) = -3 \tan x$.

$$e^{\int p(x) dx} = e^{\int -3 \tan x dx} = e^{3 \ln|\cos x|}$$

$$= \cos^3 x.$$

multiplying (3) by I F $\cos^3 x$.

$$\cos^3 x \, u' - 3u \cos^3 x \tan x = 3 \sin x \cos^2 x \cos^3 x$$

$$\cos^3 x \, u' - 3u \cos^2 x \sin x = 3 \sin x \cos^5 x$$

$$\Rightarrow \frac{d}{dx}(u \cos^3 x) = 3 \sin x \cos^5 x.$$

By integration

$$u \cos^3 x = 3 \frac{\cos^6 x}{6} + C$$

$$y^3 \cos^3 x = \frac{1}{2} \cos^6 x + C$$

(2) solve $xy' + y = xy^3$. — (1)

divide throughout by y^3

$$x\,y'/y^3 + y^{-2} = x.$$

put $y^{-2} = u$

$$u' = -2y^{-3}\,y'$$

$$u'/2 = -1/y^3\; y'$$

$$-x\,u'/2 + u = x.$$

$$u' - \frac{2u}{x} = -2 \quad \text{— (2)}$$

her $p(x) = -2/x$.

$$e^{\int p(x)\,dx} = e^{\int -2/x\,dx} = e^{-2\ln x} = e^{\ln x^{-2}} = 1/x^2$$

multiplying (2) by this integrating factor.

$$u'/x^2 - \frac{2u}{x^3} = -2/x^2$$

$$\frac{d}{dx}\left[\frac{1}{x^2}u\right] = -2/x^2$$

$$1/x^2 \, u = \int -2/x^2 \, dx$$

$$\Rightarrow u/x^2 = -2\frac{x^{-1}}{-1} + c.$$

$$u/x^2 = 2/x + c$$

$$\text{ie } \frac{1}{y^2 x^2} = 2/x + c.$$

$$1 = \frac{2x^2y^2}{x} + cx^2y^2$$

$$\underline{1 = 2xy^2 + cx^2y^2}$$