Chapter 2.

# Second order linear differential equation

A second order differential eqn is said to be linear if it can be written in the form

$y'' + p(x)y' + q(x)y = r(x)$. , If $r(x) = 0$ the equation is said to be homogeneous. Other wise it is non homogeneous. here p and 'q' are called coefficient of the equation. Now $y = \phi(x)$ is called the solution of a differential equation of second order on some interval if $\phi(x)$ is twice differentiable and equation become an identity if we replace y and its derivatives by $\phi$ and its corresponding derivatives

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eg:- $y = \cos 4x$ and $y = \sin 4x$ be the solution of the D E $y'' + 16y = 0$ —— (1)

$y = \cos 4x$

$y' = -4 \sin 4x$

$y'' = -16 \cos 4x$.

substitute $y, y', y''$ in (1) we get

$-16 \cos 4x + 16 \cos 4x$

$= 0$

ie $y = \cos 4x$ satisfies the equ$^n$ $y'' + 16y = 0$

$\therefore$ $\cos 4x$ is a soln.

Also $y = \sin 4x$.

$y'' = -16 \sin 4x$.

is also a soln of (1)

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## Fundamental Theorem or Superposition Principle or linearity Principle

If $y_1$ and $y_2$ are two solutions of the differential equation.
$L(y) = y'' + p(x) y' + q(x) y = 0$. Then the linear combination ~~$y_1$ & $y_2$ are two~~ $c_1 y_1 + c_2 y_2$ is also a solution for any values of the constants $c_1$ and $c_2$.

Proof :- Let $y_1$ and $y_2$ be any two solution of the homogeneous second order differential equation.

$$L(y) = y'' + p(x) y' + q(x) y = 0 \quad \text{--- (1)}$$

Then $L(y_1) = y_1'' + p(x) y_1' + q(x) y_1 = 0$ --- (2)

$L(y_2) = y_2'' + p(x) y_2' + q(x) y_2 = 0$ --- (3)

By substituting $y = c_1 y_1 + c_2 y_2$ and its derivatives into (1) we get.

$L(c_1 y_1 + c_2 y_2]$

$= [c_1 y_1 + c_2 y_2]'' + p(x)[c_1 y_1 + c_2 y_2]'$
$+ q(x)[c_1 y_1 + c_2 y_2]$

$= c_1 y_1'' + c_2 y_2'' + p(x) c_1 y_1' + p(x) c_2 y_2' +$
$q(x) c_1 y_1 + q(x) c_2 y_2.$

$= c_1 [y_1'' + p(x) y_1' + q(x) y_1] + c_2 [y_2'' + p(x) y_2' + q(x) y_2]$

$= c_1 \cdot 0 + c_2 \cdot 0$

$= 0.$

$\therefore c_1 y_1 + c_2 y_2$ is a solution of the given second order differential equation for any values of the constants $c_1$ & $c_2$.

Remark :- The above theorem does not hold for nonhomogeneous equation and non linear equations

Wronskian determinant

Given any two solutions $y_1$ & $y_2$ of the differential equation $y'' + p(x) y' + q(x) y = 0$, we can define a function known as Wronskian determinant or Wronskian as follows.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= y_1 y_2' - y_2 y_1'$$

eg:- If the Wronskian $W$ of $f$ and $g$ is $3e^{4t}$ and if $f(t) = e^{2t}$ find $g(t)$.

soln

$W(f, g) = 3e^{4t}$.

ie $W(f, g) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix}$

$= f(t)g'(t) - g(t)f'(t)$

$= 3e^{4t}$.

Since $f(t) = e^{2t}$, $f'(t) = 2e^{2t}$.

$\therefore W = f(t)g'(t) - g(t)f'(t) = 3e^{4t}$.

$= e^{2t} \cdot g'(t) - g(t) \times 2e^{2t} = 3e^{4t}$.

$\Rightarrow g'(t) - 2g(t) = 3e^{2t}$.

which is a first order DE of the form $y' + p(t)y = g(t)$

Then integrating factor is

$e^{\int +p(t)\,dt)} = e^{\int -2dt} = e^{-2t}$.

$\therefore$ the soln is $ye^{\int p\,dt} = \int 3e^{2t} \cdot e^{-2t}\,dt$.

$ye^{-2t} = \int 3\,dt$

$ye^{-2t} = 3t + c$.

$\therefore\; y(t) = 3te^{2t} + ce^{2t}$

eg 2:- Find the Wronskian of the following Pairs of function.

i) $e^{-2t}, te^{-2t}$.

$$W = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & te^{-2t}\times -2 + e^{-2t} \end{vmatrix}$$

$$= e^{-2t}(-2te^{-2t} + e^{-2t}) - -2e^{-2t} \cdot te^{-2t}$$

$$= -2te^{-4t} + e^{-4t} + 2te^{-4t} = e^{-4t}$$

## General solution and Fundamental set of solutions.

Let $y_1$ and $y_2$ be two solutions of the linear d.e $y'' + p(t)y' + q(t)y = 0$ — ① as long as the Wronskian of $y_1$ & $y_2$ is not everywhere zero, the linear combination $c_1y_1 + c_2y_2$ contains all solutions of ① ~~with constant~~ to the expression $y = c_1y_1 + c_2y_2$ with arbitrary constant coefficient is called general solution.

The solutions $y_1$ and $y_2$ with a nonzero Wronskian, are said to form a fundamental set of solutions of the differential equation ①.

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Theorem consider the differential equation $L(y) = y'' + p(t)y' + q(t)y = 0$ whose coefficients $p$ and $q$ are continuous on some open interval $I$, choose some point $t_0$ in $I$. Let $y_1$ be the solution of the differential equation $L(y)=0$. that also satisfies the initial conditions $y(t_0)=1$, $y'(t_0)=0$ and let $y_2$ be the solution of the differential eqn $L(y)=0$ that also satisfies the initial conditions $y(t_0)=0$, $y'(t_0)=1$. Then $y_1$ and $y_2$ form a fundamental set of solutions of $L(y)=0$.

Proof: since $p$ and $q$ are continuous on some open interval $I$ containing $t_0$, by existence and uniqueness

Theorem ∃ unique solutions $y_1$ & $y_2$ satisfying the initial value problem

$y'' + p(t)y' + q(t)y = 0$ $y(t_0) = 1$, $y'(t_0) = 0$

& $y'' + p(t)y' + q(t)y = 0$ $y(t_0) = 0$, $y'(t_0) = 1$.

The wronskian of these solutions at $t_0$ is $W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

Since wronskian of $y_1$ & $y_2$ is not zero at the point $t_0$, the functions $y_1$ & $y_2$ form a fundamental set of solutions.

pbm :- Show that $y_1(t) = e^t$ and $y_2(t) = te^t$ form a fundamental set of solutions of $y'' - 2y' + y = 0$.

Soln :- here $y_1(t)=e^t$ and $y_2(t)=te^t$

then $y_1'(t)=e^t$ & $y_2'(t)=te^t+e^t$

~~then we~~ $y_1''(t)=e^t$ & $y_2''(t)=te^t+e^t+e^t$.

Since $y_1$ is a soln $y_1$ satisfies $y''-2y'+y=0$

i.e. $e^t-2e^t+e^t=0 \Rightarrow y_1$ is a soln

and $y_2''-2y_2'+y_2 = te^t+2e^t-2(te^t+e^t)$
$+te^t$

$= te^t+2e^t-2te^t-2e^t+te^t$

$=0$

$\Rightarrow y_2$ is a soln of given d.e.

The wronskian of $y_1$ & $y_2$ is

$$W(y_1,y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^t & te^t \\ e^t & te^t+e^t \end{vmatrix}$$

$$= te^{2t}+e^{2t}-te^{2t}$$
$$= e^{2t} \neq 0.$$