Vector Calculus

1. Calculus of Multivariable Functions

Partial derivative

1 Partial derivative of a fn of several variables is its derivative with respect to one of those variables with the others held constant.

* Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(2,-1)$ if $f(x,y) = 3x^3y + 4xy^2 - 2x + 4y - 5$.

→ $F(x,y) = 3x^3y + 4xy^2 - 2x + 4y - 5$

$$\frac{\partial f}{\partial x} = 3\times 3x^2y + 4y^2 - 2 = 9x^2y + 4y^2 - 2$$

$$\left.\frac{\partial f}{\partial x}\right]_{(2,-1)} = 9\times(2)^2\times -1 + 4\times(-1)^2 - 2$$

$$= 9\times 4\times -1 + 4\times 1 - 2$$

$$= -36 + 4 - 2 = \underline{\underline{-34}}$$

$$\frac{\partial f}{\partial y} = 3x^3 + 4\times x\times 2y + 4 = 3x^3 + 8xy + 4$$

$$\left.\frac{\partial F}{\partial y}\right]_{(2,-1)} = 3\times(2)^3 + 8\times 2\times -1 + 4 = 3\times 8 - 16 + 4$$

$$= 24 - 16 + 4$$

$$= \underline{\underline{12}}$$

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* $F(x,y) = x\tan^{-1}(xy)$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$\rightarrow F(x,y) = x\tan^{-1}(xy)$

$\rightarrow \frac{\partial f}{\partial x} = \frac{1}{1+(xy)^2}(y)$

$\frac{\partial f}{\partial y} =$

* $F(x,y) = 2x^2 - 3y + \cancel{4}$

$$\frac{\partial f}{\partial x} = 2 \times 2x - 3y + \cancel{4}$$

$$= 2 \times 2x$$

$$= \underline{\underline{4x}}$$

$$\frac{\partial f}{\partial y} = \underline{\underline{-3}}$$

* $F(x,y) = x^2 - xy + y^2$

$$\frac{\partial f}{\partial x} = \cancel{2x - y}\ 2x - y$$

$$\frac{\partial f}{\partial y} = \cancel{-x +}\ \cancel{-x + 2y}\ -x + 2y$$

* $F(x,y) = \sqrt{x^2+y^2}$

$$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \times 2x = \frac{2x}{2\sqrt{x^2+y^2}}$$

$$= \frac{x}{\sqrt{x^2+y^2}}$$

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② $\frac{\partial F}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}} \times 2y = \frac{y}{\sqrt{x^2+y^2}}$

* $F(x,y) = x \tan^{-1}(xy)$

$\frac{\partial F}{\partial x} = \tan^{-1}(xy) + x \cdot \frac{\partial}{\partial x}(\tan^{-1}(xy))$

$= \tan^{-1}xy + x \cdot \frac{1}{1+(x^2y^2)} \cdot \frac{\partial}{\partial x}(xy)$

$= \tan^{-1}xy + \frac{xy}{1+(x^2y^2)}$

$\frac{\partial F}{\partial y} = x \cdot \frac{\partial}{\partial y}(\tan^{-1}xy)$

$= x \cdot \frac{1}{1+(xy)^2} \cdot \frac{\partial}{\partial y} xy$

$= \frac{x^2}{1+(xy)^2}$

Note

- $\frac{\partial F}{\partial x}$ is also denoted by f_x and $\frac{\partial f}{\partial y}$ is denoted by f_y.

* Find f_x, f_y if $f(xy) = \ln(x+2y+3z)$

→ $F(x,y,z) = \ln(x+2y+3z)$

$w = \ln(x+2y+3z)$

$\frac{\partial w}{\partial x} = \frac{1}{x+2y+3z} \times 1$

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$$\frac{\partial w}{\partial y} = \frac{1}{x+2y+3z} \times 2$$

$$\frac{\partial w}{\partial z} = \frac{1}{x+2y+3z} \times 3$$

$$f_x(x,y,z) = \frac{1}{x+2y+3z}$$

$$f_y(x,y,z) = \frac{2}{x+2y+3z}$$

$$f_z(x,y,z) = \frac{3}{x+2y+3z}$$

Limit definition in partial derivatives

$$\frac{\partial F}{\partial x} = \underset{h\to 0}{Lt} \frac{F(x+h,y)-F(x,y)}{h}$$

$$\frac{\partial F}{\partial y} = \underset{h\to 0}{Lt} \frac{F(x,y+h)-F(x,y)}{h}$$

example

* Use the limit definition of partial derivatives to complete the partial derivative of $F(x,y) = x^2+3xy+y-1$ at $(4,-5)$

→ $$\frac{\partial F}{\partial x} = \underset{h\to 0}{Lt} \frac{f(x+h,y)-f(x,y)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{f(4+h,-5)-F(4,-5)}{h}$$

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$$= \underset{h\to 0}{Lt} \frac{[(4+h)^2 + 3\times(4+h)\times -5 + (-5) - 1] - [4^2 + 3\times 4\times -5 + (-5) - 1]}{h}$$

$$= \underset{h\to 0}{Lt} \frac{[16+h^2+8h-15(4+h)-6] - [\cancel{16^2}\,16-60-6]}{h}$$

$$= \underset{h\to 0}{Lt} \frac{[h^2+8h+10-60-15h]+50}{h}$$

$$= \underset{h\to 0}{Lt} \frac{h^2-7h}{h} = \underset{h\to 0}{Lt} \frac{h^2}{h} - \frac{7h}{h} = \underset{h\to 0}{Lt}\, h-7 = 0-7$$

$$= -7$$

$$\frac{\partial f}{\partial y} = \underset{h\to 0}{Lt} \frac{f(x, y+h) - f(x,y)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{f(4,-5+h) - f(4,-5)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{[(4)^2+3\times 4\times(-5+h)-6+h] - [4^2+3\times 4\times -5 + (-5-1)]}{h}$$

$$= \underset{h\to 0}{Lt} \frac{[16+12(-5+h)-6+h] - [16-60-6]}{h}$$

$$= \underset{h\to 0}{Lt} \frac{16-60+12h-6+h-[-50]}{h}$$

$$= \underset{h\to 0}{Lt} \frac{[-50+13h]+50}{h}$$

$$= \underset{h\to 0}{Lt} \frac{13h}{h} = \underset{h\to 0}{Lt} 13$$

$$= 13$$

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* $F(x,y) = 1-x+y-3x^2y$ at $(1,2)$

$\rightarrow$ $f(x,y) = 1-x+y-3x^2y$

$$\frac{\partial f}{\partial x} = \underset{h\to 0}{Lt}\ \frac{f(x+h,y)-f(x,y)}{h}$$

$$= \underset{h\to 0}{Lt}\ \frac{f(1+h,2)-f(1,2)}{h}$$

$$= \underset{h\to 0}{Lt}\ \frac{1-(1+h)-(1+h)+2-3\times(1+h)^2\times 2-(-4)}{h}$$

$$= \underset{h\to 0}{Lt}\ \frac{1-(1+h)-(1+h)+2-(3+6h+3h^2)\times 2-(-4)}{h}$$

$$= \underset{h\to 0}{Lt}$$

$$= \underset{h\to 0}{Lt}\ \cancel{\frac{1-1-h-1-h+2-6-12h-6h^2}{h}}$$

$$= \underset{h\to 0}{Lt}\ \cancel{\frac{2-2h-1-6-12h-6h^2}{h}} = \underset{h\to 0}{Lt}$$

$$= \underset{h\to 0}{Lt}\ \frac{[h+2-6(1+2h+h^2)]+4}{h}$$

$$= \underset{h\to 0}{Lt}\ \frac{h+2-6-12h-6h^2+4}{h}$$

$$= \underset{h\to 0}{Lt}\ \frac{h+2-6-12h-6h^2+4}{h} = \underset{h\to 0}{Lt}\ \frac{-6h^2-11h}{h}$$

$$= \underset{h\to 0}{Lt}\ \frac{h(-6h-11)}{h} = \underline{\underline{-11}}$$

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④ $$\frac{\partial F}{\partial y} = \underset{h\to 0}{Lt} \frac{F(x, y+h) - F(x,y)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{F(1, 2+h) - F(1,2)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{[1-1+(2+h) - 3\times 1^2 \times (2+h)] - [1-1+2-3\times 1^2 \times 2]}{h}$$

$$= \underset{h\to 0}{Lt} \frac{2+h-6-3h+4}{h}$$

$$= \underset{h\to 0}{Lt} \frac{h-3h}{h} = \underset{h\to 0}{Lt} \frac{-2h}{h}$$

$$= \underline{\underline{-2}}$$

$$\frac{\partial F}{\partial x} = \underset{h\to 0}{Lt} \frac{F(x+h, y) - F(x,y)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{(1-(1+h)+2-3\times(1+h)^2\times 2) - [1-(1^2+2-3\times1^2\times 2)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{[1-(1+h)+2-3\times(1^2+2h+h^2)\times 2]+4}{h}$$

$$= \underset{h\to 0}{Lt} \frac{[-h+2-3(1+2h+h^2)\times 2]+4}{h}$$

$$= \underset{h\to 0}{Lt} \frac{-h+2-6-12h-6h^2+4}{h}$$

$$= \underset{h\to 0}{Lt} \frac{-6h^2-h-12h}{h} = \underset{h\to 0}{Lt} \frac{h(-6h-1-12)}{h}$$

$$= \underset{h\to 0}{Lt} -6h-13$$

$$= \underline{\underline{-13}}$$

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Partial derivatives

- $\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right)$ or $F_{xx} = (F_x)_x$

- $\frac{\partial^2 F}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right)$ or $F_{yy} = (F_y)_y$

- $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\partial F/\partial y\right)$ or $F_{yx} = (F_y)_x$

- $\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)$ or $F_{xy} = (F_x)_y$

* compute the second order partial ~~sum of~~ derivatives of the fn $g(x,y) = x^2y + \cos y + y\sin x$.

→ $g = x^2y + \cos y + y\sin x$

$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}\left(\partial g/\partial x\right) = \frac{\partial}{\partial x}\left[2xy + y\cos x\right]$$

$$= \underline{\underline{2y - y\sin x}}$$

$$\frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right) = \frac{\partial}{\partial y}\left(x^2 - \sin y + \sin x\right)$$

$$= \underline{\underline{-\cos y}}$$

$$\frac{\partial^2 g}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial}{\partial y}\left(2xy + y\cos x\right)$$

$$= \underline{\underline{2x + \cos x}}$$

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(5) $\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial g}{\partial y}\right]$

$= \frac{\partial}{\partial x}\left[x^2 - \sin y + \sin x\right]$

$= \underline{\underline{2x + \cos x}}$

* $F(x,y) = x + y + xy$

$\rightarrow \frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}[1+y]$

$= \underline{\underline{0}}$

$\frac{\partial^2 F}{\partial y^2} = \frac{\partial}{\partial y}(1+x)$

$= \underline{\underline{0}}$

$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial F}{\partial y}\right]$

$= \frac{\partial}{\partial x}[1+x]$

$= \underline{1}$

$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)$

$= \frac{\partial}{\partial y}(1+y)$

$= \underline{\underline{1}}$

* $F(x,y) = \ln(x+y)$

$\frac{\partial^2 F}{\partial x} = \frac{\partial}{\partial x}\left[\frac{\partial F}{\partial x}\right] = \frac{\partial}{\partial x}\left[\frac{1}{x+y}\right]$

$= \underline{\underline{-\frac{1}{(x+y)^2}}}$

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$$\frac{\partial^2 F}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{x+y}\right)$$

$$= -\frac{1}{(x+y)^2}$$

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{x+y}\right)$$

$$= -\frac{1}{(x+y)^2}$$

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{1}{x+y}\right)$$

$$= -\frac{1}{(x+y)^2}$$

* $h(x, y) = xe^y + y + 1$

→ $$\frac{\partial^2 h}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}(e^y)$$

$$= 0$$

$$\frac{\partial^2 h}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}(e^y x + 1)$$

$$= e^y x$$

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⑥ $$\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(e^y x + 1)$$

$$= \underline{\underline{e^y}}$$

$$\frac{\partial^2 h}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}(e^y)$$

$$= \underline{\underline{e^y}}$$

* If $z = \ln\sqrt{x^2+y^2}$, Prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

→ $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$$

$$= \underline{\underline{-\frac{x^2-y^2}{(x^2+y^2)^2}}}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial y}\right]$$

$$= \frac{\partial}{\partial y}\left[\frac{y}{x^2+y^2}\right]$$

$$= \underline{\underline{\frac{x^2-y^2}{(x^2+y^2)^2}}}$$

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$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{x^2-y^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$= 0$$

$$\therefore \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

Hence the proof.

* show that $w = 5\cos(3x+3ct) + e^{x+ct}$ satisfies the wave equation $\frac{\partial^2 w}{\partial t^2} = \frac{c^2 \partial^2 w}{\partial x^2}$

→ $w = 5\cos(3x+3ct) + e^{x+ct}$

$$\frac{\partial w}{\partial t} = *15c\sin(3x+3ct) + e^{x+ct}c$$

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t}\right)$$

$$= \frac{\partial}{\partial t}\left(15c\sin(3x+3ct) + e^{x+ct}c\right)$$

$$= -45c^2 \sin 3x+3ct + e^{x+ct}c^2$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(e^{x+ct} + 15\cos(3x+3ct)\right)$$

$$= -45\sin(3x+3ct) + e^{x+ct}$$

$$c^2 \times \frac{\partial^2 w}{\partial x^2} = -45c^2\sin(3x+3ct) + e^{x}+ct\, c^2$$

$$\therefore \frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

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If $F(x,y)$ and it's partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined and continues in (a,b) then

$$F_{xy}(a,b) = F_{yx}(a,b)$$

The partial differential equation $\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0$ is known as two-dimensional Laplace equation.

The three-dimensional Laplace eqn is $\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = 0$.

Differentiability

A fn $f(x,y)$ is differentiable at (x_0, y_0) if $f_x(x_0, y_0)$ and $F_y(x_0, y_0)$ exist and it satisfies the eqn

$$\Delta z = f_x(x_0,y_0)\Delta x + F_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\Delta z = f_x(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$ and $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$

Theorem

If a fn $f(x,y)$ is differentiable at (x_0, y_0) the f is continues at (x_0, y_0)

Proof

Let $f(x,y)$ is differentiable at (x_0, y_0). Then by definition its partial derivatives $f_x(x_0, y_0)$ and $F_y(x_0, y_0)$

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is exist and it satisfies the eqn.

$$\Delta z = f_x(x_0, y_0) + f_y(x_0, y_0) + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \quad -(1)$$

where $\Delta z = f_x(x_0 y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ and

$\varepsilon_1 + \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$.

Let (x, y) be any point arbitrarily close to (x_0, y_0) and eqn (1) becomes

$$f(x, y) - f(x_0, y_0) = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

Hence $f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

If $\Delta x, \Delta y$ approach to zero then $f(x, y)$ approaches $f(x_0, y_0)$ Hence f is continues at (x_0, y_0).

Chain Rule

Let $w = f(x)$ be a differentiable fn of x and let $x = g(t)$ be a differentiable fn of t. Then

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{dx} \cdot \frac{dx}{dt}}$$

This is the 'chain rule' for functions of a single variable.

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Chain rule for functions of two variables

Let $w = f(x,y)$ be a fn of two variables x and y and let $x = \phi(t)$ and $y = \psi(t)$. Then,

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt}}$$

Chain rule for three variables.

Let $w = f(x,y,z)$ is differentiable and x, y and z are differentiable functions of t, then w is a differentiable function of t and $\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}}$

Chain rule for two independent variables and three intermediate variables

Let $w = f(x,y,z)$, $x = g(r,s)$, $y = h(r,s)$ and $z = k(r,s)$. Then,

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}$$

Chain rule for two independent variables and 2 I.M.V

If $w = f(x,y)$, $x = g(r,s)$, $y = h(r,s)$ then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s}$$

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* Use the chain rule to find the derivative of $w = x^2 + y^2$ with respect to t along the path $x = \cos t$, $y = \sin t$. What is the derivative's value at $t = \pi$?

→ $w = x^2 + y^2$, $x = \cos t$; $y = \sin t$

By chain rule,

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} \qquad \text{—(1)}$$

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2) = 2x = \underline{\underline{2\cos t}}$$

$$\frac{dx}{dt} = \frac{d}{dt}(\cos t) = \underline{\underline{-\sin t}}$$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}[x^2 + y^2] = 2y = \underline{\underline{2\sin t}}$$

$$\frac{dy}{dt} = \frac{d}{dt}[\sin t] = \cos t$$

$$(1) \Rightarrow \frac{dw}{dt} = 2\cos t \times -\sin t + 2\sin t \times \cos t$$

$$= -2\cos t \sin t + 2\sin\cos t$$

$$= \underline{\underline{0}}$$

Also $\left.\frac{dw}{dt}\right]_{t=\pi} = 0$ //

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* $W = xy$

$x = \cos t,\quad y = \sin t \quad \text{at } t = \pi$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{dw}{dy}\cdot\frac{dy}{dt}$$

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(xy) = y = \sin t$$

$$\frac{dx}{dt} = \frac{d}{dt}(\cos t) = -\sin t$$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial x}(xy) = x = \cos t$$

$$\frac{dy}{dt} = \frac{d}{dt}(\sin t) = \cos t$$

$$\frac{dw}{dt} = -\sin t(\sin t) + \cos t(\cos t)$$

$$= -0 + 1 = 1$$

* $W = x^2 + y^2,\quad x = \cos t + \sin t,\quad y = \cos t - \sin t \quad \text{at } t = 0.$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt}$$

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2) = 2x$$

$$= 2(\cos t + \sin t) = 2\cos t + 2\sin t$$

$$\frac{dx}{dt} = \frac{d}{dt}(\cos t + \sin t) = -\sin t + \cos t$$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial x}(x^2+y^2) = 2y$$

$$= 2\cos t - 2\sin t$$

$$\frac{dy}{dt} = \frac{d}{dt}(\cos t - \sin t)$$

$$= -\sin t - \cos t$$

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$$\frac{dw}{dt} = (2\cos t + 2\sin t)\cdot(-\sin t + \cos t) + (2\cos t - 2\sin t)\cdot(-\sin t - \cos t)$$

$$\left.\frac{dw}{dt}\right]_{t=0} = 2(2\times 1) + (2\times -1)$$

$$= 2 - 2 = \underline{\underline{0}}$$

* $w = xy + z,\ x = \cos t,\ y = \sin t,\ z = t$ at $t = 0$

→ $\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(xy+z) = y = \sin t$

$$\frac{dx}{dt} = \frac{d}{dt}(\cos t) = -\sin t$$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}(xy+z) = x = \cos t$$

$$\frac{dy}{dt} = \frac{d}{dt}(\sin t) = \cos t$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}(xy+z) = 1$$

$$\frac{dz}{dt} = \frac{d}{dt}(t) = 1$$

$$\frac{dw}{dt} = -\sin^2 t + \cos^2 t + 1$$

$$\left.\frac{dw}{dt}\right]_{t=0} = -\sin^2(0) + \cos^2(0) + 1$$

$$= 0 + 1 + 1$$

$$= \underline{\underline{2}}$$

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4 $w = z - \sin xy, \quad x = t, \quad y = \ln t, \quad z = e^{t-1}$ at $t = 1$

$$\rightarrow \frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(z - \sin xy)$$

$$= -y\cos(xy) = -\ln t \cos(\ln t(t))$$

$$\frac{dx}{dt} = \frac{d}{dt}(t) = 1$$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}(z - \sin xy)$$

$$= -x\cos(xy) = -t\cos(\ln t(t))$$

$$\frac{dy}{dt} = \frac{d}{dt}\ln t$$

$$= 1/t$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}(z - \sin xy)$$

$$= 1$$

$$\frac{dz}{dt} = \frac{d}{dt}(e^{t-1})$$

$$= e^{t-1}$$

$$\frac{dw}{dt} = \cancel{-y\cos(xy) \cdot 1 + -x\cos(xy)(1/t)} + e^{t-1}$$

$$\left.\frac{dw}{dt}\right]_{t=1} = -\ln t \cos((\ln t)(t)) - t\cos((\ln(t))(t)) \times 1/t + e^{t-1}$$

$$= -\ln(1)\cos(\ln(1)(1)) - \cos(\ln(1)(1)) + e^{0}$$

$$= 0 - 1 + 1 = 0$$

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* using the chain rule express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s.

$w = x + 2y + z^2, \quad x = r/s, \quad y = r^2 + \ln s, \quad z = 2r$

$\rightarrow \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$

$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(x + 2y + z^2)$

$= \underline{\underline{1}}$

$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(r/s) = \underline{\underline{1/s}}$

$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}(x + 2y + z^2) = \underline{\underline{2}}$

$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(r^2 + \ln s) = \underline{\underline{2r}}$

$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}(x + 2y + z^2) = 2z$

$2z = 2(2r) = \underline{\underline{4r}}$

$\frac{\partial z}{\partial r} = \frac{\partial}{\partial r}(2r) = \underline{\underline{r}}$

$\frac{\partial w}{\partial r} = 1 \cdot 1/s + 2 \cdot 2r + 4r \times r$

$= \underline{\underline{1/s + 4r + 4r^2}}$

$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}$

$\frac{\partial x}{\partial s} = \frac{\partial}{\partial s}(r/s) = -r/s^2$

$\frac{\partial y}{\partial s} = \frac{\partial}{\partial s}(r^2 + \ln s) = 1/s$

$\frac{\partial z}{\partial s} = \frac{\partial}{\partial s}(2r) = 0$

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(11)

$$\frac{\partial w}{\partial s} = 1\cdot(-8/s^2) + 2\times 1/s + 4s\times 0$$

$$\frac{\partial w}{\partial s} = -8/s^2 + 2/s$$

$$\frac{\partial w}{\partial s} = 2/s - 8/s^2$$

* Using chain rule express $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ in terms of u and v if $w = xy+yz+zx$, $x=u+v$, $y=u-v$ and $z=uv$. Also evaluat $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ at $(1/2, 1)$

→ $w = xy+yz+zx$

$x = u+v$, $y=u-v$, $z=uv$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$$

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(xy+yz+zx)$$
$$= y+z = (u-v)+uv$$

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}(u+v) = 1$$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}(xy+yz+zx)$$
$$= x+z = (u+v)+uv$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}(u-v) = 1$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}(xy+yz+zx) = x+y = (u+v)+(u-v)$$
$$= 2u$$

$$\frac{\partial z}{\partial u} = \frac{\partial}{\partial u}(uv) = v$$

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$$\frac{\partial w}{\partial u} = ((u-v)+uv)\cdot 1 + [(u+v)+uv]\cdot 1 + 2u(v)$$

$$\left.\frac{\partial w}{\partial u}\right]_{(1/2,1)} = ((1/2-1)+1/2) + [(1/2+1)+1/2] + 2\,1/2(1)$$

$$= 0 + 2 + 1$$

$$= \underline{\underline{3}}$$

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}(u+v)$$

$$= \underline{\underline{1}}$$

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}(u-v)$$

$$= \underline{\underline{-1}}$$

$$\frac{\partial z}{\partial v} = \frac{\partial}{\partial v}(uv)$$

$$= \underline{\underline{u}}$$

$$\frac{\partial w}{\partial v} = (u-v+uv)\cdot 1 + [(u+v)+uv]\cdot -1 + 2u(u)$$

$$= [(u-v)+uv] - [(u+v)+uv] + 2u^2$$

$$= u-v+uv-(u+v)-uv+2u^2$$

$$= u-v-u-v+2u^2$$

$$= -2v+2u^2$$

$$\left.\frac{\partial w}{\partial v}\right]_{(1/2,1)} = -2(1)+2(1/2)^2$$

$$= -2+2/4$$

$$= -2+1/2$$

$$= \underline{-3/2}$$

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(12) $\frac{\partial w}{\partial u} = ((u-v)+uv)\cdot 1 + ((u+v)+uv)\cdot 1 + 2u(v)$

$= (u-v)+uv + u+v+uv + 2uv$

$= 2u + 4uv$

$\left.\frac{\partial w}{\partial u}\right|_{(1/2,1)} = 2(1/2) + 4 \times 1/2 \times 1$

$= 1 + 2 = 3$

* Using chain rule express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$ in terms of r and θ. If $w = \tan^{-1}(y/x)$, $x = r\cos\theta$, $y = r\sin\theta$

at $(1, \pi/6)$

→ $w = \tan^{-1} y/x$

$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \cancel{\frac{\partial w}{\partial z}}$

$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\cancel{(\tan^{-1}(y/x))}$

$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\tan^{-1}(y/x)$

$= -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = -\frac{\sin\theta}{r}$

$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}(\tan^{-1}(y/x))$

$= \frac{1}{x + y^2/x} = \frac{1}{r\cos\theta + \frac{r^2\sin^2\theta}{r\cos\theta}} = \frac{1}{r\cos\theta + r\sin\theta\tan\theta}$

$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(r\cos\theta) = \cos\theta$

$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(r\sin\theta) = \sin\theta$

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$$\frac{\partial w}{\partial r} = -\frac{\sin\theta}{r} \times \cos\theta + \frac{1}{r[\cos\theta + \sin\theta\tan(\theta)]} \times \sin\theta$$

$$= \frac{-\cos\theta\sin\theta}{r} + \frac{\sin\theta}{r(\cos\theta + \sin\theta\tan\theta)}$$

$$\left.\frac{\partial w}{\partial r}\right]_{(1,\pi/6)} = \frac{-\cos\pi/6\,\sin(\pi/6)}{1} + \frac{\sin\pi/6}{(\cos\pi/6 + \sin\pi/6\tan\pi/6)}$$

$$= -\sqrt{3}/2 \times 1/2 + \frac{1/2}{\sqrt{3}/2 + 1/2 \times \sqrt{3}/3}$$

$$= -\sqrt{3}/4 + \sqrt{3}/4$$

$$= \underline{\underline{0}}$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$\frac{\partial x}{\partial \theta} = \frac{\partial}{\partial x}(r\cos\theta)$$

$$= -r\sin\theta$$

$$\frac{\partial y}{\partial \theta} = \frac{\partial}{\partial \theta}(r\sin\theta)$$

$$= r\cos\theta$$

$$\frac{\partial w}{\partial \theta} = -\frac{\sin\theta}{r} \times (-r\sin\theta) + \frac{r\cos\theta}{r(\cos\theta + \sin\theta\tan\theta)}$$

$$= \frac{\cancel{r}\sin^2\theta}{\cancel{r}} + \frac{\cos\theta}{\cos\theta + r\sin\theta\tan\theta}$$

$$= \sin^2\theta + \frac{\cos\theta}{\cos\theta + \sin\theta\tan\theta}$$

$$\left.\frac{\partial w}{\partial \theta}\right]_{(1,\pi/6)} = 1/4 + \frac{\sqrt{3}/2}{\sqrt{3}/2 + 1/2 \times \sqrt{3}/3}$$

$$= 1/4 + 3/4 = \underline{\underline{1}}$$

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(13)

Implicit Differentiation using chain rule.

Let $w = f(x, y)$ be a differentiable fn of two variables then, $\frac{dy}{dw} = \frac{-F_x}{F_y}$, where $F_x = \frac{\partial F}{\partial x}$ and $F_y = \frac{\partial F}{\partial y}$

* Find $\frac{dy}{dx}$ if $xe^{y} + \sin xy + y - \ln 2 = 0$

→ $F(x, y) = xe^{y} + \sin xy + y - \ln 2$

$F_x = \frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(xe^{y} + \sin xy + y - \ln 2)$

$F_x = e^{y} + y\cos xy$

∴ $F_y = \frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(xe^{y} + \sin xy + y - \ln 2)$

$F_y = xe^{y} + x\cos xy + 1$

∴ $\frac{dy}{dx} = \frac{-F_x}{F_y} = -\frac{[e^{y} + y\cos xy]}{[xe^{y} + x\cos xy + 1]}$

* Find $\frac{dy}{dx}$ at the given point

1. $F(x,y) = x^3 - 2y^2 + xy = 0$; $(1, 1)$

→ $\frac{dy}{dx} = -\frac{F_x}{F_y}$

$F_x = \frac{\partial F}{\partial x} = \frac{\partial}{\partial x}[x^3 - 2y^2 + xy]$

$F_x = 3x^2 + y$

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$$F_y = \frac{\partial}{\partial y}(x^3 - 2y^2 + xy)$$

$$F_y = x - 4y$$

$$\frac{dy}{dx} = -\left(\frac{3x^2+y}{x-4y}\right)$$

$$\left.\frac{dy}{dx}\right]_{(1,1)} = -\frac{3+1}{1-4} = \frac{-4}{-3}$$

$$= -\frac{4}{-3} = \frac{4}{3}$$

2) $F(x,y) = xy + y^2 - 3x - 3 = 0 \quad ; (-1,1)$

$\rightarrow$ $$F_x = \frac{\partial}{\partial x}(xy + y^2 - 3x - 3)$$

$$= y - 3$$

$$F_y = \frac{\partial}{\partial y}(xy + y^2 - 3x - 3)$$

$$= x + 2y$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{y-3}{x+2y}$$

$$\left.\frac{dy}{dx}\right]_{(-1,1)} = -\frac{1-3}{-1+2} = 2$$

3) $F(x,y) = x^2 + xy + y^2 - 7 = 0 \quad ; (1,2)$

$\rightarrow$ $$F_x = \frac{\partial}{\partial x}(x^2 + xy + y^2 - 7)$$

$$= 2x + y$$

$$F_y = \frac{\partial}{\partial y}(x^2 + xy + y^2 - 7) = x + 2y$$

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(14)

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x+y}{x+2y}$$

$$\left.\frac{dy}{dx}\right]_{(1,2)} = -\frac{2+2}{1+4} = -\frac{4}{5}$$

4) $F(x,y) = x^2+\sin y-2y$; $(2,0)$

$\rightarrow$ $F_x = \frac{\partial}{\partial x}(x^2+\sin y-2y)$

$= \cancel{2}\ 2x$

$F_y = \frac{\partial}{\partial y}(x^2+\sin y-2y)$

$= -2+\cos(y)$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x}{-2+\cos y}$$

$$\left.\frac{dy}{dx}\right]_{(2,0)} = -\frac{2(2)}{-2+\cos(0)} = -\frac{4}{-2+1}$$

$= 4$

5) $F(x,y) = 1-x-y^2-\sin xy = 0$; $(0,1)$

$F_x = \frac{\partial}{\partial x}(1-x-y^2-\sin xy)$

$= -y\cos xy - 1$

$F_y = \frac{\partial}{\partial y}(1-x-y^2-\sin xy)$

$= -x\cos xy - 2y$

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$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-y\cos xy - 1}{-x\cos xy - 2y}$$

$$\left.\frac{dy}{dx}\right]_{(0,1)} = -\frac{-1\cos(0)(1) - 1}{-(0)\cos(0)(1) - 2(1)}$$

$$= -\frac{-2}{-2}$$

$$= \underline{\underline{-1}}$$

6) $F(x,y) = 2xy + e^{x+y} - 2 = 0$; $(0, \ln 2)$

$\rightarrow$ $F_x = \frac{\partial}{\partial x}(2xy + e^{x+y} - 2)$

$= \underline{\underline{2y + e^{x+y}}}$

$F_y = \frac{\partial}{\partial y}(2xy + e^{x+y} - 2)$

$= \underline{\underline{2x + e^{x+y}}}$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2y + e^{x+y}}{2x + e^{x+y}}$$

$$\left.\frac{dy}{dx}\right]_{(0,\ln 2)} = -\frac{2\ln 2 + e^{0+\ln 2}}{2(0) + e^{0+\ln 2}}$$

$$= -\frac{\ln(4) + 2}{2}$$

$$= \underline{\underline{-\frac{\ln(4) + 2}{2}}} = \underline{\underline{-\ln(2) + 1}}$$

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15

Linearization

The linearization of a fn $F(x,y)$ at (x_0,y_0) is the fn

~~$L(x,y) = f(x_0,y_0) + f_x(x-x_0) + f_y(x_0,y_0)$~~

$$L(x,y) = F(x_0,y_0) + F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0)$$

* Find the linearization of $F(x,y) = e^x \cos y$ at $(0, \pi/2)$?

→ $F(x,y) = e^x \cos y$, $x_0 = 0$, $y_0 = \pi/2$

$$L(x,y) = F(x_0,y_0) + F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0) \quad \text{—(1)}$$

$$F(x_0,y_0) = F(0,\pi/2) = e^0 \cos(\pi/2) = 1 \times 0 = 0$$

$$F_x = e^x \cos y$$

$$F_x(x_0,y_0) = F_x(0,\pi/2) = e^0 \cos(\pi/2) = 1 \times 0 = 0$$

$$F_y = -e^x \sin y$$

$$F_y(x_0,y_0) = F_y(0,\pi/2) = -e^0 \sin(\pi/2) = -1 \times 1 = -1$$

$\therefore$ eqn (1) $\Rightarrow$

$$L(x,y) = 0 + 0 \times (x-0) - 1 \times (y - \pi/2)$$

$$= -(y - \pi/2)$$

$$L(x,y) = \pi/2 - y$$

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1. $F(x,y) = x^2+y^2+1$ at a) $(0,0)$, b) $(1,1)$

$\rightarrow$ $L(x,y) = F(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

a) $F(x_0,y_0) = F(0,0) = 0+0+1 = \underline{\underline{1}}$

$f_x(x_0,y_0) = f_x(0,0)$

$f_x = 2x$

$f_x(0,0) = \underline{\underline{0}}$

$f_y(0,0) = 2y = \underline{\underline{0}}$

$\therefore L(x,y) = 1 + 0\times(x-0) + 0\times(y-0)$

$L(x,y) = \underline{\underline{1}}$

b) $f_x(1,1) = 1^2+1^2+1$

$= \underline{\underline{3}}$

$f_x = 2\times 1 = \underline{\underline{2}}$

$f_y = 2\times 1 = \underline{\underline{2}}$

$\therefore L(x,y) = 3 + 2\times(x-1) + 2\times(y-1)$

$= 3 + 2x - 2 + 2y - 2$

$L(x,y) = \underline{\underline{2x+2y-1}}$

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(18)

2. $F(x,y) = 3x-4y+5$ at a) $(0,0)$ b) $(1,1)$

$\rightarrow$ a) $L(x,y) = F(x_0,y_0) + F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0)$

$F(x_0,y_0) = F(0,0)$

$= 3(0)-4(0)+5$

$= 5$

$F_x = 3$

$F_y = -4$

$L(x,y) = 5 + 3(x-0) - 4(y-0)$

$= 5 + 3x - 4y$ $= \underline{3x-4y+5}$

b) $F(x_0,y_0) = F(1,1)$

$= 3(1) - 4(1) + 5$

$\cancel{= 3+4+5 = 12}$ $3-4+5 \underline{= 4}$

$F_x = 3$

$F_y = -4$

$L(x,y) = 4 + 3(x-1) - 4(y-1)$

$= 4 + 3x - 3 - 4y + 4$

$L(x,y) = \cancel{3x+4y}$ $\underline{3x-4y}+3$

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3: $F(x,y) = e^x \cos y$ at (a). $(0,0)$ b) $(0,\pi)$

→ $F(x,y) = e^x \cos y$

a) $L(x,y) = F(x_0, y_0) + f_x(x_0 y_0)(x - x_0) + f_y(x_0 y_0)(y - y_0)$

$F(x_0 y_0) = F(0,0)$

$= e^0 \cos 0$

$= 1 \times 1 = \underline{\underline{1}}$

$f_x(x_0 y_0) = e^x \cos y$

$f_x(0,0) = e^0 \cos 0 = \underline{\underline{1}}$

$f_y(x_0 y_0) = -e^x \sin y$

$f_y(0,0) = -e^0 \sin(0)$

$= \underline{\underline{0}}$

$L(x,y) = 1 + 1 \times (x - 0) + 0 \times (y - 0)$

$= \underline{\underline{1 + x}}$

b) $F(x_0, y_0) = F(0, \pi)$

$= e^0 \cos \pi = 1 \times -1$

$= \underline{\underline{-1}}$

$f_x(0,\pi) = e^0 \cos \pi = \underline{\underline{-1}}$

$f_y(0,\pi) = -e^0 \sin \pi = \underline{\underline{0}}$

$L(x,y) = -1 + -1 \times (x - 0) + 0(y - \pi)$

$= \underline{\underline{-1 - x}}$

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(57)

4) $f(x,y) = x^3y^4$ at a) $(0,0)$ b $(1,1)$

$\rightarrow$ $L(x,y) = f(x_0 y_0) + f_x(x_0 y_0)(x - x_0) + f_y(x_0 y_0)(y - y_0)$

a) $f(x_0 y_0) = 0^3 \times 0^4 = 0$

$f_x(0,0) = 3x^2y^4$

$= \underline{0}$

$f_y(0,0) = x^3 4y^3$

$= \underline{\underline{0}}$

$L(x,y) = 0 + 0 \times (x-0) + 0(y-0)$

$= \underline{\underline{0}}$

b) $f(x_0 y_0) = f(1,1) = 1^3 \times 1^4$

$= \underline{\underline{1}}$

$f_x(1,1) = 3 \times 1^2 \times 1^4$

$= 3$ //

$f_y(1,1) = 1^3 \times 4 \times 1^3$

$= 4$ //

$L(x,y) = 1 + 3(x-1) + 4(y-1)$

$= 1 + 3x - 3 + 4y - 4$

$= \underline{\underline{3x + 4y - 6}}$

5. $f(x,y) = (x+y+2)^2$ at a) $(0,0)$, b) $(1,2)$

a) $f(x_0 y_0) = (0+0+2)^2 = 4$ //

$f_x(0,0) = 2(x+y+2)$

$= 2(0+0+2) = 4$ //

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$f_y(0,0) = 2(0+0+2) = 4$

$L(x,y) = 4 + 4(x-0) + 4(y-0)$

$= 4 + 4x + 4y$

$= 4(x+y+1)$

b) $F(x_0,y_0) = (1+2+2)^2 = 25$

$f(x) = 2(x+y+2).$

$= 2(1+2+2) = 10$

$f_y = 2(1+2+2)$

$= 10$

$L(x,y) = 25 + 10(x-1) + 10(y-2)$

$= 25 + 10x - 10 + 10y - 20$

$= \cancel{25}\, 10x + 10y - 5$

Error in the standard Linear approximation

If F has continuous first and second partial derivatives and M is any upper bound for the values of $|f_{xx}| = |f_{yy}|$ and $|f_{xy}|$ then the error $E(x,y)$ satisfies the inequality

$$|E(x,y)| \leq \frac{1}{2} M \left[|x-x_0| + |y-y_0|\right]^2$$

* Find the linearization of $f(x,y) = 1 + y + x\cos y$ at $(0,0)$

Also find the error, where $|x| \leq 0.2$, $|y| \leq 0.2$?

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(18) → $f(x,y) = 1+y+x\cos y$, $x_0 = 0$, $y_0 = 0$

$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$f(x_0,y_0) = f(0,0) \Rightarrow 1+0+0\times\cos 0 = \underline{\underline{1}}$

$f_x = \cos y$

$f_x(x_0,y_0) = f_x(0,0) = \cos 0 = 1$

$f_y = 1 + x \times -\sin y = 1 - x\sin y$

$f_y(x_0,y_0) = f_y(0,0) = 1 - 0\sin(0)$

$= \underline{\underline{1}}$

$\therefore L(x,y) = 1 + 1\times(x-0) + 1\times(y-0)$

$= \underline{\underline{1+x+y}}$

$f_x = \cos y$

$f_{xx} = \underline{0}$

$|f_{xx}| = |0| = \underline{\underline{0}}$

$f_y = 1 - x\sin y$

$f_{yy} = 0 - x\cos y = -x\cos y$

$|f_{yy}| = |1 - x\cos y| = |-x||\cos y|$

$= |x||\cos y|$

$|f_{yy}| \leq 0.2 \times 1$

$|f_{yy}| \leq \underline{0.2}$

$f_{xy} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}(1 - x\sin y) = -\sin y$

$|f_{xy}| = |-\sin y| = |\sin y| \leq 1$

Take the largest value of $|f_{xx}|, |f_{yy}|, |f_{xy}|$ as the value of M $\therefore$ M = 1

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$\therefore$ Error is

$$|\epsilon(x,y)| \le \frac{1}{2} \times 1 \times [|x-0|+|y-0|]^2$$
$$\le \frac{1}{2}[|x|+|y|]^2$$
$$\le \frac{1}{2}[0.2+0.2]^2$$
$$\le \frac{1}{2}[0.4]^2$$
$$\le \frac{1}{2} \times 0.16$$
$$\le \underline{\underline{0.088}}$$

* Find the linearization and error of fn $f(x,y)$ at P_0.

$\Rightarrow$ i) $f(x,y) = x^2 - 3xy + 5$ at $P_0(2,1)$; $R: |x-2| \le 0.1, |y-1| \le 0.1$

$\rightarrow L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$$f(x_0,y_0) = f(2,1)$$
$$= 2^2 - 3\times2\times1+5$$
$$= 4-6+5 = \underline{\underline{3}}$$

$$f_x(x_0,y_0) = f_x(2,1)$$
$$= 2x-3y = 2(2)-3(1)$$
$$= 4-3 = \underline{\underline{1}}$$

$$f_y(x_0,y_0) = f_y(2,1) = -3x$$
$$= -3(2) = \underline{\underline{-6}}$$

$$L(x,y) = 3 + 1\times(x-2) + -6\times(y-1)$$
$$= 3 + x - 2 - 6y + 6$$
$$= \underline{\underline{x - 6y + 7}}$$

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(19)

$f_{xx} = 2$

$|f_{xx}| = |2| = 2$

$f_{yy} = 0$

$|f_{yy}| = |0| = \underline{\underline{0}}$

$f_{xy} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = -3$

$|f_{xy}| = |-3| = \underline{\underline{3}}$

$M = 3$

$\therefore$ Error is

$|E(x,y)| = \frac{1}{2} \times 3\left[|x-2| + |y-1|\right]^2$

$= \frac{1}{2} \times 3 \left[0.1 + 0.1\right]^2$

$= \frac{3}{2} \times 0.04$

$= \underline{\underline{1.53}}$

* $F(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$ at $P_0(3,2)$ R: $|x-3| \leq 0.1$

$|y-2| \leq 0.1$

$\Rightarrow F(x_0, y_0) = F(3,2)$

$= 3^2 - 3\times 2 + \frac{1}{2}(2)^2 + 3$

$= 9 - 6 + 2 + 3 = \underline{\underline{8}}$

$f_x(x_0, y_0) = f_x(3,2)$

$= 2x - y = 2(3) - 2$

$= 6 - 2 = \underline{\underline{4}}$

$f_y(x_0, y_0) = f_y(3,2) = -x + y$

$= -3 + 2 = \underline{\underline{-1}}$

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$L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0)$

$= 8 + 4(x-3) + (-1)(y-2)$

$= 8 + 4x - 12 - y + 2$

$= \underline{\underline{4x - y - 2}}$

$|f_{xx}| = |2| = \underline{\underline{2}}$

$|f_{yy}| = |1| = |1|$

$|f_{xy}| = |-1| = 1$

$\therefore M = 2$

error is,

$|E(x,y)| = \frac{1}{2} \times 2 \left[|x-3| + |y-2|\right]^2$

$= [0.1 + 0.1]^2$

$= [0.2]^2 = \underline{\underline{0.04}}$

* $f(x,y) = \sin x \cos y$, $P_0(\pi/4, \pi/4)$, $R: |x-\pi/4| \le 0.1, |y-\pi/4| \le 0.1$

$\Rightarrow L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0)$

$f(x_0, y_0) = f(\pi/4, \pi/4)$

$= \sin\pi/4 \times \cos\pi/4 =$

$= \sqrt{2}/2 \times \frac{\sqrt{2}}{2} = 2/4 = \underline{\underline{1/2}}$

$f_x(\pi/4, \pi/4) = \cos\pi/4 \cos\pi/4$

$= \underline{\underline{1/2}}$

$f_y(\pi/4, \pi/4) = -\sin\pi/4 \sin\pi/4$

$= -1/2$

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$f_{xx} = -\cos y \sin x$

$|f_{xx}| = |-\cos\pi/4 \sin\pi/4|$

$= |\cos\pi/4 \sin\pi/4| = \underline{\underline{1/2}}$

$|f_{yy}| = |-\cos y \sin x| = |\cos\pi/4 \sin\pi/4|$

$= \underline{\underline{1/2}}$

$f_{xy} = -\cos x \sin y$

$|f_{xy}| = |-\cos\pi/4 \sin\pi/4| = |\cos\pi/4 \sin\pi/4|$

$= \underline{\underline{1/2}}$

$\therefore M = 1/2$

$\therefore$ error is

$E(x,y) = \frac{1}{2} \times \frac{1}{2} \left[|x-\pi/4| + |y-\pi/4|\right]^2$

$= \frac{1}{4} \times [0.1 + 0.1]^2$

$= \frac{1}{4} \times 0.04 = \underline{\underline{0.01}}$

~~$L(x,y) =$~~

Function of more than two variables.

1) The Linearization of $f(x,y,z)$ at a point $P_0(x_0,y_0,z_0)$ is

$$L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$$

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2) The error $\epsilon(x,y,z)$ satisfies the inequality,

$$|\epsilon(x,y,z)| \le \frac{1}{2}M\left[|x-x_0|+|y-y_0|+|z-z_0|\right]^2$$

* Find the linearization $L(x,y,z)$ of $f(x,y,z) = xz-3yz+2$ at the point $(x_0,y_0,z_0)=(1,1,2)$. Find an upper bound for the error accured in replacing f by L on the region

$R: |x-1|\le 0.01,\ |y-1|\le 0.01,\ |z-2|\le 0.02$

→ $f(x,y,z) = xz-3yz+2$

$$L(x,y,z) = f(x_0,y_0,z_0)+f_x(x_0,y_0,z_0)(x-x_0)+f_y(x_0,y_0,z_0)(y-y_0)$$
$$+f_z(x_0,y_0,z_0)(z-z_0)$$

$f(x_0,y_0,z_0)= f(1,1,2)$

$= (1\times 2)-(3\times 1\times 2)+2$

$= 2-6+2 = \underline{\underline{-2}}$

$f_x(x_0,y_0,z_0) = f_x(1,1,2)$

$= z = \underline{\underline{2}}$

$f_y(1,1,2) = -3z = (-3)\times 2$

$= -6$

$f_z(1,1,2) = x-3y$

$= 1-3 = \underline{\underline{-2}}$

$L(x,y,z) = -2+2(x-1)-6(y-1)-2(z-2)$

$= -2+2x-2-6y+6-2z+4$

$= \underline{2x-6y-2z+6}$

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(2)

$|f_{xx}(x_0, y_0, z_0)| = |0| = 0$

$|f_{yy}(x_0, y_0, z_0)| = |0| = 0$

$|f_{zz}(x_0, y_0, z_0)| = |0| = 0$

$|f_{xy}(x_0, y_0, z_0)| = |0| = 0$

$|f_{xz}| = |0| = 0$

$|f_{yz}| = |3| = 3$

$\therefore M = 3$.

$$E(x, y, z) \leq \frac{1}{2} \times M \left[|x - x_0| + |y - y_0| + |z - z_0|\right]^2$$

$$\cancel{\leq \frac{1}{2} \times 0 \times \left[|x-1| + |y-1| + |z-2|\right]^2}$$

$$\cancel{\leq 0}$$

$$\leq \frac{1}{2} \times 3 \times \left[|x-1| + |y-1| + |z-2|\right]^2$$

$$\leq \frac{3}{2} \times \left[0.01 + 0.01 + 0.02\right]^2$$

$$\leq \frac{3}{2} \times (0.04)^2$$

$$\leq \frac{3}{2} \times 0.0016$$

$$\leq \underline{\underline{0.0024}}$$

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Differential

If we move from (x_0, y_0) to a point (x_0+dx, y_0+dy) the resulting change $df = f_x(x_0, y_0)dx + f_y(x_0, y_0)\,dy$ in the linearization of f is called the "total differential" or "differential" of f.

- Absolute change $\Rightarrow df$
- Relative change $\Rightarrow \dfrac{df}{f(x_0, y_0)}$
- Percentage change $\Rightarrow \dfrac{df}{f(x_0, y_0)} \times 100$

* suppose a cylandrical can is designed to have a radius of 1 inch and a height of 5 inch, but that the radius and height are off by the amounts $dr = 0.03$ and $dh = -0.1$, estimate the resulting absolute, relative and percentage changes in the volume of the can?

→ $r = 1$ inch

$h = 5$ inch

$dr = 0.03$

$dh = -0.1$

volume of cylinder $(V) = \pi r^2 h$

i) absolute change $= dV = V_r(r,h)\,dr + V_h(r,h)\,dh$ —(1)

$$V_r = 2\pi r h$$

$$V_r(r,h) = V_r(1,5) = 2\times\pi\times 1 5$$

$$= \underline{10\pi}$$

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(22)

$V_h = \pi r^2$

$V_h(r,h) = V_h(1,5) = \pi(1)^2$

$= \pi$

$\therefore$ Absolute change $= 10\pi \times 0.03 + \pi \times -0.1$

$= 0.3\pi - 0.1\pi$

$= 0.2\pi$

ii) relative change $= \dfrac{dv}{V(r,h)}$ — (1)

$V(r,h) = V(1,5) = \pi(1)^2 \times 5 = 5\pi$

$\therefore$ (1) $\Rightarrow$ relative change $= \dfrac{0.2\pi}{5\pi}$

$= \dfrac{0.2}{5} = 0.04$

iii) Percentage change $= \dfrac{dv}{V(r,h)} \times 100$

$= 0.04 \times 100$

$= 4\%$

Partial Derivatives with constrained variables

Consider a function, $w = f(x,y,z)$ Then,

$\left(\dfrac{\partial w}{\partial x}\right)_y$ denotes $\dfrac{\partial w}{\partial x}$ with x and y independent.

$\left(\dfrac{\partial F}{\partial y}\right)_{xt}$ denotes $\dfrac{\partial F}{\partial y}$ with y, x and t independent.

* If $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$, evaluate

i) $\left(\dfrac{\partial w}{\partial x}\right)_y$ ii) $\left(\dfrac{\partial w}{\partial x}\right)_z$ iii) $\left(\dfrac{\partial w}{\partial y}\right)_z$ iv) $\left(\dfrac{\partial w}{\partial z}\right)_y$

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i) $W = x^2 + y^2 + z^2$ $\qquad \because z = x^2 + y^2$

$W = x^2 + y^2 + [x^2 + y^2]^2$

$W = x^2 + y^2 + x^4 + y^4 + 2x^2y^2$

$\cancel{W = x^2 + y^2 + x^4 + y^4 +}$

$$\left[\frac{\partial W}{\partial x}\right]_y = 2x + 4x^3 + 4xy^2$$

ii) Here we differentiate W partially w.r.to x considering x and z as independent. so y is dependent. eliminating y from the eqn by replacing

y^2 by $z - x^2$

$\therefore W = x^2 + y^2 + z^2 = x^2 + (z - x^2) + z^2$ $\qquad \because z = x^2 + y^2$, $y^2 = z - x^2$

$W = z + z^2$

$$\left(\frac{\partial W}{\partial x}\right)_z = 0$$

iii) Here differentiat W w.r.t to y considering y and z as independent variables and x as dependent. So we eliminate from the equation by replacing x^2 by $z - y^2$

$\therefore W = x^2 + y^2 + z^2 = z - y^2 + y^2 + z^2$ $\qquad \left[\because z = x^2 + y^2,\ x^2 = z - y^2\right]$

$W = z + z^2$

$$\left(\frac{\partial W}{\partial y}\right)_z = 0$$

iv) Differentiate W w.r.to z considering z and y as independent variables and x as dependent so we eliminate x from the eqn by replacing

x^2 by $z - y^2$

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$\therefore W = x^2+y^2+z^2$

$= (z-y^2)+y^2+z^2$

$= z+z^2$

$\left(\frac{\partial w}{\partial z}\right)_y = 1+2z$

* • If $w=x^2+y-z+\sin t$ and $x+y=t$ find $\left(\frac{\partial w}{\partial x}\right)_{y,z}$?

→ $w=x^2+y-z+\sin t$

$\therefore w = x^2+y-z+\sin(x+y)$

$\left(\frac{\partial w}{\partial x}\right)_y = 2x+\cos(x+y)$

* $W=x^2y^2+yz-z^3$ and $x^2+y^2+z^2=6$ Find $\left(\frac{\partial w}{\partial y}\right)_z$

→ $W= x^2y^2+yz-z^3$

$x^2+y^2+z^2=6$

$y^2 = 6-(x^2+z^2)$

$\therefore W = x^2(6-(x^2+z^2)) + 6-x^2-z^2-z^3$

$= x^2(6-x^2-z^2)+6-x^2-z^2-z^3$

$= 6x^2-x^4-x^2z^2+6-x^2-z^2-z^3$

* $w = x^2y^2 + yz - z^3$ and $x^2+y^2+z^2=6$ find $\left(\frac{\partial w}{\partial y}\right)_z$

→ $w = x^2y^2+yz-z^3$

$x^2+y^2+z^2=6$

$x^2 = 6-y^2-z^2$

$\therefore w = (6-z^2-y^2)y^2+yz-z^3$

$= 6y^2-y^4-y^2z^2+yz-z^3$

$\left(\frac{\partial w}{\partial z}\right)_z = 12y-4y^3-2yz^2+z$

* $w = x^2+y-z+\sin t$, and $x+y=\sin t$

$t = x+y$

i) $\left(\frac{\partial w}{\partial y}\right)_{x,z}$

$w = x^2+y-z+\sin(x+y)$

$\left(\frac{\partial w}{\partial y}\right)_{x,z} = 1+\cos(x+y)$

$x = t-y$

ii) $w = (t-y)^2+y-z+\sin t = t^2-2ty+y^2-z+\sin t$

$\left(\frac{\partial w}{\partial y}\right)_{z,t} = -2t+2y+1$

$= 1-2t+2y$

iv) $\left(\frac{\partial w}{\partial z}\right)_{y,t}$

$w = (t-y)^2+y-z+\sin t$

$\left(\frac{\partial w}{\partial z}\right)_{y,t} = -1$

vi) $\left(\frac{\partial w}{\partial t}\right)_{y,z}$

$w = (t-y)^2+y-z+\sin t = t^2-2ty+y^2-y-z+\sin t$

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$\left(\frac{\partial w}{\partial t}\right)_{y,z} = 2t - 2y + \cos t$

iii) $w = x^2 + y - z + \sin(x+y)$

$\left(\frac{\partial w}{\partial z}\right)_{x,y} = -1$

~~v) $w = x^2 + y - z + \sin t$~~

~~$y = t - x$~~

~~$\therefore w = x^2 + (t-x) - z + \sin t$~~

~~$\left(\frac{\partial w}{\partial t}\right)_{x,z} = 1 + \cos t$~~

v) $w = x^2 + (t-x) - z + \sin t$

$y = t - x$

$\left(\frac{\partial x}{\partial t}\right)_{x,z} = 1 + \cos t$

Directional Derivative

The derivative of F at $P_0(x_0, y_0)$ in the direction of the unit vector $u = u_1\hat{i} + u_2\hat{j}$ is the number.

$$\left(\frac{dF}{ds}\right)_{u,P_0} = \underset{s\to 0}{Lt} \frac{F(x_0 + su_1, y_0 + su_2) - F(x_0, y_0)}{s}$$

It is denoted by $(D_u F)_{P_0}$

* Find the derivative of $f(x,y) = 2xy - 3y^2$ at $P_0(5,5)$ in the direction of the vector $4\hat{i} + 3\hat{j}$.

→ unit vector $\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{4i + 3\hat{j}}{\sqrt{4^2 + 3^2}} = \frac{4i + 3\hat{j}}{5}$

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$\hat{u} = \frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}$ (here $u_1 = 4/5$, $u_2 = 3/5$)

Directional derivative $= \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$

$$= \lim_{s \to 0} \frac{f(5 + s \times 4/5,\ 5 + s \times 3/5) - f(5, 5)}{s}$$

$$= \lim_{s \to 0} \frac{\left[2(5 + s \times 4/5)(5 + \frac{3s}{5}) - 3(5 + 3s/5)^2\right] - (2 \times 5 \times 5 - 3 \times 5^2)}{s}$$

$$= \lim_{s \to 0} \frac{\left[2(25 + 7s + \frac{12s^2}{25}) - 3(25 + 6s + \frac{9s^2}{25})\right] - [50 - 75]}{s}$$

$$= \lim_{s \to 0} \frac{\left(50 + 14s + \frac{24s^2}{25} - 75 - 18s - \frac{27s^2}{25} + 25\right)}{s}$$

$$= \lim_{s \to 0} \frac{-4s - \frac{3s^2}{25}}{s}$$

$$= \lim_{s \to 0} \frac{-4s}{s} - \frac{3s^2}{25/s}$$

$$= \lim_{s \to 0} -4 - \frac{3s}{25}$$

$$= -4 - \frac{3(0)}{25} = -4$$

* ~~using the~~ Find the directional derivative

i) $f(x, y) = x^2 + xy$ at $(1, 2)$ direction $i + j$

→ $\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{\hat{i} + \hat{j}}{\sqrt{1^2 + 1^2}} = \frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}}$

$u_1 = 1/\sqrt{2}$, $u_2 = 1/\sqrt{2}$

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$$D.D = \underset{s\to 0}{Lt} \frac{f(x_0+su_1, y_0+su_2)-f(x_0,y_0)}{s}$$

$$= \underset{s\to 0}{Lt} \frac{f(1+s\frac{1}{\sqrt{2}}, 2+s\frac{1}{\sqrt{2}}) - f(1,2)}{s}$$

$$= \underset{s\to 0}{Lt} \frac{\left[(1+\frac{s}{\sqrt{2}})^2 + (1+\frac{s}{\sqrt{2}})(2+\frac{s}{\sqrt{2}})\right] - (1^2+1\times 2)}{s}$$

$$= \underset{s\to 0}{Lt} \frac{\left[(1+\frac{s^2}{2}+\frac{2s}{\sqrt{2}}) + 2 + \frac{2s}{\sqrt{2}} + \frac{s}{\sqrt{2}} + \frac{s^2}{2}\right] - 3}{s}$$

$$= \underset{s\to 0}{Lt} \frac{3 + \frac{2s^2}{2} + \frac{5s}{\sqrt{2}} - 3}{s}$$

$$= \underset{s\to 0}{Lt} \frac{s^2 + \frac{5s}{\sqrt{2}}}{s}$$

$$= \underset{s\to 0}{Lt} \frac{s^2}{s} + \frac{5s}{\sqrt{2}\,s}$$

$$= \underset{s\to 0}{Lt}\ s + 5/\sqrt{2}$$

$$= 0 + 5/\sqrt{2} = 5/\sqrt{2}$$

* $f(x,y) = 2x^2+y^2$, $P_0(-1,1)$, $A = 3i-4j$

$$\rightarrow \hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{3\hat{i}-4\hat{j}}{\sqrt{(3^2+(-4)^2)}} = \frac{3}{5}i - \frac{4}{5}\hat{j}$$

$$D.D = \underset{s\to 0}{Lt} \frac{f(-1+s\,3/5,\ 1+s\,4/5) - f(-1,1)}{s}$$

$$= \underset{s\to 0}{Lt} \frac{\left[2(-1+s\,3/5)^2 + (1+4s/5)^2\right] - (2\times(-1)^2+1^2)}{s}$$

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$$= \underset{s\to 0}{Lt} \frac{\left[2\left(1+\frac{6s}{5}+\frac{9s^2}{25}\right)+\left(1+\frac{8s}{5}+\frac{16s^2}{25}\right)\right]-3}{s}$$

$$= \underset{s\to 0}{Lt} \frac{2+\frac{12s}{5}+\frac{18s^2}{25}+1+\frac{8s}{5}+\frac{16s^2}{25}-3}{s}$$

$$= \underset{s\to 0}{Lt} \frac{\left(-\frac{20s}{5}+\frac{34s^2}{25}\right)}{s}$$

$$= \underset{s\to 0}{Lt} -\frac{20}{5}+\frac{34s}{25}$$

$$= \frac{34\times 0}{25}-\frac{20}{5}.$$

$$= \underline{\underline{-4}}$$

Gradient vector

The gradient vector $f(x,y)$ at a point $P_0(x_0,y_0)$ is

$$\nabla f = \frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}$$

* Find the gradient of $f(x,y)=y-x$ at $(2,1)$

→ $f(x,y)=y-x$

$\frac{\partial f}{\partial x} = -1,\ \frac{\partial f}{\partial y} = 1$

$\nabla f = -1\hat{i}+1\hat{j}$

$\nabla f\big|_{(2,1)} = \underline{\underline{-1\hat{i}+1\hat{j}}}$

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1) $f(x,y) = \ln(x^2+y^2)$ at $(1,1)$

$\rightarrow \dfrac{\partial f}{\partial x} = \dfrac{2x}{x^2+y^2}$

$\dfrac{\partial f}{\partial y} = \dfrac{2y}{x^2+y^2}$

$\nabla f = \dfrac{\partial f}{\partial x}\hat{i} + \dfrac{\partial f}{\partial y}\hat{j}$

$= \dfrac{2x}{x^2+y^2}\hat{i} + \dfrac{2y}{x^2+y^2}\hat{j}$

$\nabla f(1,1) = \dfrac{2}{2}i + \dfrac{2}{2}\hat{j}$

$= \underline{\hat{i}+\hat{j}}$

2. $g(x,y) = y - x^2$ at $(-1,0)$

$\rightarrow \dfrac{\partial g}{\partial x} = -2x$

$\dfrac{\partial g}{\partial y} = 1$

$\nabla f = \dfrac{\partial g}{\partial x}\hat{i} + \dfrac{\partial g}{\partial y}\hat{j}$

$= -2x\hat{i} + 1\hat{j}$

$\Delta f_{(-1,0)} = -2(-1)\hat{i} + \hat{j}$

$= \underline{2\hat{i}+\hat{j}}$

3. $g(x,y) = x^2/2 - y^2/2$ at $(\sqrt{2}, 1)$

$\rightarrow \dfrac{\partial g}{\partial x} = x, \ \dfrac{\partial g}{\partial y} = -y$

$\nabla f = \dfrac{\partial g}{\partial x}i + \dfrac{\partial g}{\partial y}$

$= x\hat{i} - y\hat{j}$

$\nabla f(\sqrt{2},1) = \underline{\sqrt{2}\hat{i} - \hat{j}}$

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4- ~~$f(x,y,z) = 2z^3 - 3(x^2+y^2$~~

4- $f(x,y,z) = x^2+y^2-2z^2+z\ln x$ at $(1,1,1)$

$\rightarrow$ $\frac{\partial f}{\partial x} = 2x + z/x$

$\frac{\partial f}{\partial y} = 2y$, $\frac{\partial f}{\partial z} = -4z + \ln x$

$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$= (2x+z/x)\hat{i} + 2y\hat{j} + (-4z+\ln x)\hat{k}$

$\nabla f_{(1,1,1)} = (2+1)\hat{i} + 2\hat{j} + (-4+\ln 1)\hat{k}$

$= \underline{3\hat{i} + 2\hat{j} - 4\hat{k}}$

5. $f(x,y,z) = 2z^3 - 3(x^2+y^2)z + \tan^{-1}xz$ at $(1,1,1)$

$\rightarrow$ $\frac{\partial f}{\partial x} = \frac{-6zx(1+x^2z^2) - z}{1+x^2z^2}$

$\frac{\partial f}{\partial y} = -6yz$

$\frac{\partial f}{\partial z} = \frac{-6z^2(1+x^2z^2) - x}{1+x^2z^2}$

$\nabla f = \left(\frac{-6zx(1+x^2z^2)-z}{1+x^2z^2}\right)\hat{i} + (-6yz)\hat{j} + \left(\frac{-6z^2(1+x^2z^2)-x}{1+x^2z^2}\right)\hat{k}$

$\nabla f_{(1,1,1)} = \left(\frac{-6(1)(1)(1+1)-1}{1+1}\right)\hat{i} + (-6)\hat{j} + \left(\frac{-6(1)(1+1)-1}{1+1}\right)\hat{k}$

$= -\frac{11}{2}\hat{i} - 6\hat{j} - \frac{11}{2}\hat{k} = \underline{-\left(\frac{11}{2}\hat{i} + 6\hat{j} + \frac{11}{2}\hat{k}\right)}$

6. $f(x,y,z) = (x^2+y^2+z^2)^{-1/2} + \ln(xyz)$ at $(-1,2,-2)$

$\rightarrow$ $\frac{\partial f}{\partial x} = -\frac{x}{(x^2+y^2+z^2)^{3/2}} + 1/x$

$\frac{\partial f}{\partial y} = -\frac{y}{(x^2+y^2+z^2)^{3/2}} + \frac{1}{y}$

$\frac{\partial f}{\partial z} = -\frac{z}{(x^2+y^2+z^2)^{3/2}} + 1/z$

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$$\nabla f = \left(-\frac{x}{(x^2+y^2+z^2)^{3/2}} + \frac{1}{x}\right)\hat{i} + \left(-\frac{y}{(x^2+y^2+z^2)^{3/2}} + \frac{1}{y}\right)\hat{j} + \left(-\frac{z}{(x^2+y^2+z^2)^{3/2}} + \frac{1}{z}\right)\hat{k}$$

$$\nabla f_{(-1,2,-2)} = \left(\frac{1}{((-1)^2+(2)^2+(2)^2)^{3/2}} + \frac{1}{-1}\right)\hat{i} + \left(-\frac{2}{[(-1)^2+2^2+(-2)^2]^{3/2}} + \frac{1}{2}\right)\hat{j} + \left(\frac{2}{((-1)^2+2^2+(-2)^2)^{3/2}} + \frac{1}{-2}\right)\hat{k}$$

$$= -2\hat{i} + \frac{5}{2}\hat{j} - \frac{5}{2}\hat{k}$$

$$\nabla f_{(-1,2,-2)} = -\left(\left[\frac{1}{27} + \frac{1}{-1}\right]\hat{i} + \left[\frac{2}{27} + \frac{1}{2}\right]\hat{j} + \left[\frac{-2}{27} + \frac{1}{-2}\right]\hat{k}\right)$$

$$= -\left(-\frac{26}{27}\hat{i} + \frac{31}{54}\hat{j} - \frac{31}{54}\hat{k}\right)$$

$$= \frac{26}{27}\hat{i} - \frac{31}{54}\hat{j} + \frac{31}{54}\hat{k}$$

D.D =

3 $f(x,y) = \cos x \cos y$, $P_0(\pi/4, \pi/4)$, $A = 3\hat{i}+4\hat{j}$

$$\Rightarrow \hat{u} = \frac{3\hat{i}+4\hat{j}}{5}$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

$$\frac{\partial f}{\partial x} = -\cos y \sin x$$

$$\frac{\partial f}{\partial y} = -\cos x \sin y$$

$$\nabla f\Big]_{(\pi/4,\pi/4)} = \left(-\cos\tfrac{\pi}{4}\sin\tfrac{\pi}{4}\right)\hat{i} + \left(-\cos\tfrac{\pi}{4}\sin\tfrac{\pi}{4}\right)\hat{j}$$

$$= -\tfrac{1}{2}\hat{i} - \tfrac{1}{2}\hat{j} = -\left(\tfrac{1}{2}\hat{i} + \tfrac{1}{2}\hat{j}\right)$$

$$D.D = -\left(\tfrac{1}{2}\hat{i} + \tfrac{1}{2}\hat{j}\right)\cdot\frac{(3\hat{i}+4\hat{j})}{5} = \frac{-3/2 - 2}{5} = \frac{-7}{10}$$

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4. $f = xy + yz + zx \quad , P_0(1,-1,2)$

$A = 3i + 6j - 2k$

$\rightarrow \hat{u} = \dfrac{3i + 6\hat{j} - 2k}{\sqrt{3^2 + 6^2 + (-2)^2}} = \dfrac{3\hat{i} + 6\hat{j} + \hat{k}}{7}$

$\dfrac{\partial f}{\partial x} = y + z$

$\dfrac{\partial f}{\partial y} = x + z \;, \; \dfrac{\partial f}{\partial z} = y + x$

$\nabla f \Big]_{(1,-1,2)} = (1+2)\hat{i} + (1+2)\hat{j} + (-1+1)\hat{k}$

$= i + 3\hat{j}$

$D \cdot D = (i + 3\hat{j}) \cdot \dfrac{3i + 6\hat{j} - 2k}{7}$

$= \dfrac{3 + 18}{7} = \dfrac{21}{7} = \underline{\underline{3}}$

5. $f = x^2 + 2y^2 - 3z^2 , \quad P_0(1,1,1) , A = i + j + k$

$\hat{u} = \dfrac{i + \hat{j} + k}{\sqrt{3}}$

$\dfrac{\partial f}{\partial x} = 2x , \; \dfrac{\partial f}{\partial y} = 4y , \; \dfrac{\partial f}{\partial z} = 6z$

$\nabla f \Big)_{(1,1,1)} = 2\hat{i} + 4\hat{j} - 6\hat{k}$

$D \cdot D = (2i + 4\hat{j} + 6\hat{k}) \cdot \dfrac{i + j + k}{\sqrt{3}}$

$= \dfrac{2 + 4 + 6}{\sqrt{3}} = \cancel{\dfrac{12}{\sqrt{3}} = 4\sqrt{3}}$

$= {}^{0}/_{\sqrt{3}} = \underline{\underline{0}}$

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Remark

Directional interms of gradient at P_o is

Directional derivative $= (\nabla f)_{P_o} \cdot \vec{u}$

* Find the derivative of $f(x,y) = xe^y + \cos(xy)$ at $P_o(2,0)$ in the direction of $\hat{i}+\hat{j}$

$\rightarrow$ $\hat{u} = \frac{\hat{i}+\hat{j}}{\sqrt{1^2+1^2}} = \frac{\hat{i}+\hat{j}}{\sqrt{2}}$

$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$ — (1)

$\frac{\partial f}{\partial x} = e^y - \sin(xy) \times y = e^y - y\sin xy$

$\frac{\partial f}{\partial y} = xe^y - x\sin(xy)$

$\therefore$ (1) $\Rightarrow \nabla f = [e^y - y\sin(xy)]\,\hat{i} + [xe^y - x\sin(xy)]\,\hat{j}$

$(\nabla f)_{P_o} = \Delta f\big]_{(2,0)} = (e^0 - 0)\hat{i} + [2\times e^0 - 2\sin(2\times 0)]$

$= \hat{i} + [2 - 2\sin 0]\hat{j}$

$= \underline{\underline{\hat{i} + 2\hat{j}}}$

$\therefore$ Directional Derivative $= (\hat{i}+2\hat{j}) \cdot \frac{(\hat{i}+\hat{j})}{\sqrt{2}}$

$= \frac{1+2}{\sqrt{2}} = \underline{\underline{3/\sqrt{2}}}$

1 $f(x,y) = 2xy - 3y^2$, $P_o(5,5)$, $A = 4\hat{i} + 3\hat{j}$

$\rightarrow$ $\hat{u} = \frac{4\hat{i}+3\hat{j}}{\sqrt{4^2+3^2}} = \frac{4\hat{i}+3\hat{j}}{5}$

$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$

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$$\frac{\partial f}{\partial x} = 2y, \quad \frac{\partial f}{\partial y} = 2x - 6y$$

$$\nabla f = 2y\,\hat{i} + (2x - 6y)\,\hat{j}$$

$$(\nabla f)_{P_0} = \nabla f(5,5)$$

$$= 2(5)\hat{i} + (2(5) - 6(5))\hat{j}$$

$$= \underline{\underline{10i - 20\hat{j}}}$$

Directional derivative $= \dfrac{4\hat{i} + 3\hat{j}}{5} \cdot 10\hat{i} - 20\hat{j}$

$$= \cancel{40} \; \frac{40 - 60}{5} =$$

$$= -\frac{20}{5} = \underline{\underline{-4}}$$

$f = 2x^2 + y^2, \; P_0(-1,1), \; A = 3\hat{i} - 4\hat{j}$

$$\hat{u} = \frac{3\hat{i} - 4\hat{j}}{\sqrt{4^2 + 3^2}} = \frac{3\hat{i} - 4\hat{j}}{5}$$

$$\frac{\partial f}{\partial x} = 4x, \quad \frac{\partial f}{\partial y} = 2y$$

$$\nabla f\big]_{P_0} = \nabla f\big]_{(-1,1)}$$

$$= 4(-1)\hat{i} + 2(1)\hat{j}$$

$$= -4\hat{i} + 2\hat{j}$$

$$D.D = \frac{3\hat{i} - 4\hat{j}}{5} \cdot -4\hat{i} + 2\hat{j}$$

$$= \frac{-12 - 8}{5} = \frac{-20}{5} = \underline{-4}$$

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(29)

6. $g = e^{x}\cos(yz)$, $P_0(0,0,0)$, $A = 2i + j - 2k$

$\rightarrow \hat{u} = \dfrac{2i + \hat{j} - 2k}{\sqrt{9}} = \dfrac{2i + j - 2k}{3}$

$\dfrac{\partial g}{\partial x} = 3\cos(yz) \cdot e^{x}$, $\dfrac{\partial g}{\partial y} = 3e^{x} - \sin yz \cdot xz$

$\dfrac{\partial g}{\partial z} = 3e^{x} - \sin yz \cdot y$

$\nabla f\Big]_{(0,0,0)} = [3 \times e^{0} \cdot \cos(0\times 0)]\hat{i} + [3\times 0 \cdot e^{0} - \sin(0,0)]\hat{j} + [3e^{0} \cdot 0 - \sin(0\cdot 0)]\hat{k}$

$= (3 \times 1 \times 1)\hat{i} - 3i$

$D \cdot D = 3i \cdot \dfrac{2i + 2j - 2k}{3}$

$= \dfrac{6}{3} = \underline{\underline{2}}$

7. $F = x^{3} - xy^{2} - z$ $\quad P_0(1,1,0)$ $\quad A = 2i - 3j + 6k$

$\rightarrow \hat{u} = \dfrac{2i - 3\hat{j} + 6k}{\sqrt{49}} = \dfrac{2i - 3\hat{j} + 6k}{7}$

$\dfrac{\partial f}{\partial x} = 3x^{2} - y^{2}$, $\dfrac{\partial f}{\partial y} = -2xy$, $\dfrac{\partial f}{\partial z} = -1$

$\nabla f\Big]_{(1,1,0)} = (3 \times 1^{2} - 1^{2})i + (-2 \times 1 \times 1)\hat{j} + -k$

$= \underline{\underline{2\hat{i} - 2\hat{j} - k}}$

$D \cdot D = 2\hat{i} - 2\hat{j} - \hat{k} \cdot \dfrac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7}$

$= \dfrac{4 + 6 - 6}{7} = \underline{\underline{4/7}}$

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Tangents of level curves

Let $f(x,y)$ be a function of two variables. Then the equation to tangent lines to level curves is

$$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)=0$$

* Find an eqn for the tangent to the circle $x^2+y^2=4$ at the point $(0,-2)$

→ $f(x,y)=x^2+y^2$

$f_x=2x$

$f_x(x_0,y_0)=f_x(0,-2)=2\times 0=\underline{\underline{0}}$

$f_y=2y$

$f_y(x_0,y_0)=f_y(0,-2)$

$=2\times(-2)=\underline{\underline{-4}}$

∴ Eqn of tangent to the circle is

$0\times(x-0)+-4(y+2)=0$

$-4(y+2)=0$

$-4y-8=0$

$-4y=8$

$y=\frac{8}{-4}=\underline{\underline{-2}}$

* $f(x,y)=x^2+y^2=4$, $(\sqrt{2},\sqrt{2})$

→ $f_x=2x$, $f_y=2y$

$f_x(x_0,y_0)=f_x(\sqrt{2},\sqrt{2})$

$=\underline{\underline{2\sqrt{2}}}$

$f_y(\sqrt{2},\sqrt{2})=\underline{\underline{2\sqrt{2}}}$

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30.

Eqn of tangent to the circle is

$2\sqrt{2}(x-\sqrt{2}) + 2\sqrt{2}(y-\sqrt{2}) = 0$

$2\sqrt{2}x - 4 + 2\sqrt{2}y - 4 = 0$

$2\sqrt{2}x + 2\sqrt{2}y - 8 = 0$

$2\sqrt{2}(x+y) = 8$

$(x+y) = \frac{8}{2\sqrt{2}} = 4/\sqrt{2}$

$= \underline{2\sqrt{2}}$

$y = \underline{-x+2\sqrt{2}}$

2. $(x^2 - y) = 1$, $(\sqrt{2}, 1)$

$\rightarrow$ $f_x(\sqrt{2},1) = 2x = 2\sqrt{2}$

$f_y(\sqrt{2},1) = -1$

Eqn of tangent

$2\sqrt{2}(x-\sqrt{2}) + (-1)(y-1) = 0$

$2\sqrt{2}x - 4 - y + 1 = 0$

$\underline{2\sqrt{2}x - y = 3}$

3. $xy = -4$, $(2,-2)$

$\rightarrow$ $f_x(2,-2) = y = -2$

$f_y(2,-2) = x = 2$

Eqn of tangent

$-2(x-2) + 2(y+2) = 0$

$-2x+4+2y+4 = 0$

$-2x+2y = -8$

$-x+y = -4$

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$$y = x - 4$$

4. $\frac{x^2}{4} + y^2 = 2$, $(-2, 1)$

→ $f_x = \frac{2x}{4} = \frac{x}{2}$

$= -\frac{2}{2} = -1$

$f_y\Big]_{(-2,1)} = 2y = 2 \times 1 = 2$

$-1(x+2) + 2(y-1) = 0$

$-x - 2 + 2y - 2 = 0$

$-x + 2y = 4$

$$y = \frac{4+x}{2}$$

Equations for tangent planes and normal lines

The eqn to ther tangent plane of the level surface $f(x, y, z) = c$ at the point P_0 is

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

The eqn to the ~~tange~~ normal line of the level surface $f(x, y, z) = c$ at the point P_0 is,

$$x = x_0 + f_x(x_0, y_0, z_0)t, \quad y = y_0 + f_y(x_0, y_0, z_0)t, \quad z = f_z(x_0, y_0, z_0)t$$

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(31)

* Find the tangent Plane and normal line of the surface

$f(x,y,z) = x^2+y^2+z^2-3$ at P_0 $(1,1,1)$

$f_x = 2x$

$f_x(x_0 y_0 z_0) = f_x(1,1,1)$

$= 2$

$f_y = 2y$

$f_y(1,1,1) = 2\times 1 = 2$

$f_z(1,1,1) = 2z = 2$

Eqn to the tangent plane is,

$2(x-1)+2(y-1)+2(z-1)=0$

$2x-2+2y-2+2z-2=0$

$2x+2y+2z-6=0$

$2(x+y+z-3)=0$

$x+y+z-3=0$

$\underline{\underline{x+y+z=3}}$

The line normal to the surface at $(1,1,1)$ is,

$x = x_0 + f_x(x_0 y_0 z_0)t$

$x = \underline{\underline{1+2t}}$

$y = y_0 + f_y(x_0, y_0, z_0)t$

$= \underline{\underline{1+2t}}$

$z = z_0 + f_z(x_0, y_0, z_0)t$

$= \underline{\underline{1+2t}}$

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1) $x^2+y^2+z=9$, $P_0(1,2,4)$

$\rightarrow$ $f_x(x_0,y_0,z_0) = f_x(1,2,4)$

$= 2x = 2\times 1$

$= \underline{\underline{2}}$

$f_y(x_0,y_0,z_0) = f_x(1,2,4)$

$= 2y = 2\times 2 = \underline{\underline{4}}$

$f_z(x_0,y_0,z_0) = f_z(1,2,4)$

$= \underline{\underline{1}}$

Eqn. of the tangent plane is

$2(x-1)+4(y-2)+1(z-4) = 0$

$2x-2+4y-8+z-4 = 0$

$2x+4y+z-14 = 0$

$\underline{\underline{2x+4y+z = 14}}$

The normal line to the surface is,

$x = 1+2t$

$y = 2+4t = 2(1+2t)$

$z = 4+t$

2. $x^2+y^2-z^2=18$, $P_0(3,5,-4)$

$\rightarrow$ $f_x(3,5,-4) = 2x = 6$

$f_y(3,5,-4) = 2y = 2\times 5 = 10$

$f_z(3,5,-4) = -2z = -2\times -4$

$= \underline{\underline{8}}$

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(32) Eqn of the tangent plane is,

$6(x-3)+10(y-5)+8(z+4)=0$

$6x-18+10y-50+8z+32=0$

$6x+10y+8z-36=0$

$2(3x+5y+4z-18)=0$

$\underline{\underline{3x+5y+4z=18}}$

The line normal to the surface is,

$x = 3+6t = 3(1+2t)$

$y = 5+10t = 5(1+2t)$

$z = -4+8t = 2(-1+2t)$

3. $2z-x^2=0$, $P_0(2,0,2)$

$\Rightarrow f_x(2,0,2) = -2x$

$= -2\times 2 = \underline{\underline{-4}}$

$f_y(2,0,2) = \underline{\underline{0}}$

$f_z(2,0,2) = \underline{\underline{2}}$

Eqn of the tangent plane

$-4(x-2)+0(y-0)+2(z-2)=0$

$-4x+8+0+2z-4=0$

$2z-4x+4=0$

$2(z-2x+2)=0$

$z-x+2=0$

$\underline{z-x=-2}$

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The normal line to the surface is

$x = 2-4t \quad = 2(1-2t)$

$y = 0+0t \quad = 0$

$z = 2+2t \quad = 2(1+t)$

4. $\cos\pi x - \cancel{2}x^2y + e^{xz} + yz = 0 \qquad P_0(0,1,2)$

$\Rightarrow f_x(0,1,2) = -\pi \sin\pi x - 2xy + e^{xz} z$

$= -\pi \sin\pi(0) - 2(0)(1) + e^{0(2)} \times 2$

$= 0 - 0 + 2 = \underline{\underline{2}}$

$f_y(0,1,2) = -x^2 + z = 0^2 + 2$

$= 2$

$f_z(0,1,2) = e^{xz} x + y$

$= e^{0(2)} \times \cancel{2}\, 0 + 1$

$= \underline{\underline{1}}$

Eqn of the tangent is,

$2(x-0) + 2(y-1) + (z-2) = 0$

$2x + 2y - 2 + z - 2 = 0$

$2x + 2y + z - 4 = 0$

$\underline{\underline{2x+2y+z=4}}$

Line normal to the surface,

$x = 2t$

$y = 1+2t$

$z = 2+t$

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(33)

5. $x^2+y^2-2xy-x+3y-z=-4$, $P_0(2,-3,18)$

$\Rightarrow$ $f_x = 2x-2y-1$

$= 2\times2-(2\times-3)-1$

$= \underline{\underline{9}}$

$f_y(2,-3,18) = 2y-2x+3$

$= \underline{\underline{-7}}$

$f_z = -1$

eqn of the tangent.

$9(x-2)+(-7)(y+3)+-1(z-18)=0$

$9x-18-7y-21-z+18=0$

$9x-7y-z-21=0$

$\underline{\underline{9x-7y-z=21}}$

normal line is,

$x = 2+9t$

$y = -3-7t$

$z = 18-t$

6. $x^2-y-5z=0$; $P_0(2,-1,1)$

$\rightarrow$ $f_x = 2x = 4$

$f_y = -1$

$f_z = -5$

eqn of tengent is,

$4(x-2)+-1(y+1)-5(z-1)=0$

$4x-8-y-1-5z+5=0$

$\underline{\underline{4x-y-5x = 4}}$

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Eqn of the normal line is,

$x = 2 + 4t = 2(1+2t)$

$y = -1 - t$

$z = 1 - 5t$

* Show that gradient is orthogonal to velocity vector.

→ Proof

If $r = g(t)i + h(t)j + k(t)k$ is a smooth curve on the level surface $f(x,y,z) = c$ then,

$f(x,y,z) = c \Rightarrow f(g(t), h(t), k(t)) = c$ — (1)

Differentiating both sides of (1) with respect to t leads to

$$\frac{d}{dt}[f(g(t), h(t), k(t))] = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\cdot\frac{dg}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dh}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dk}{dt} = 0 \qquad \text{(by chain rule)}$$

$$\underbrace{\left(\frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k\right)}_{\nabla f}\cdot\underbrace{\left(\frac{dg}{dt}i + \frac{dh}{dt}j + \frac{dk}{dt}k\right)}_{dr/dt} = 0$$

$$\nabla f\cdot\frac{dr}{dt} = 0 \qquad\qquad \frac{dr}{dt} = \vec{v}$$

$$\nabla f\cdot\vec{v} = 0$$

$\therefore \nabla f$ is orthogonal to the velocity vector.

[$\therefore$ two vectors $\vec{a}$ and $\vec{b}$ said to be orthogonal if $\vec{a}\cdot\vec{b} = 0$]

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(34)

* Find the parametric eqns for the line tangent to the curve of intersection of the surfaces $x^3+3x^2y^2+y^3+4xy-z^2=0$, $x^2+y^2+z^2=11$ at $(1,1,3)$

$\Rightarrow$ Let $f(x,y,z)=x^3+3x^2y^2+y^3+4xy-z^2$

$g(x,y,z)=x^2+y^2+z^2$

$\nabla f=\frac{\partial f}{\partial x}i+\frac{\partial f}{\partial y}j+\frac{\partial f}{\partial z}k$

$=(3x^2+6xy^2+4y)i+(6x^2y+3y^2+4x)j+(-2z)k$

$\nabla f)_{(1,1,3)}=(3\times1^2+6\times1\times1^2+4\times1)i+(6\times1^2\times1+3\times1^2+4\times1)j+(-2\times3)k$

$=(3+6+4)i+(6+3+4)j-6k$

$=13i+13j-6k$

$\nabla g=\frac{\partial g}{\partial x}i+\frac{\partial g}{\partial y}j+\frac{\partial g}{\partial z}k$

$\nabla g]_{(1,1,3)}=(2\times1)i+(2\times1)j+(2\times3)k$

$=2i+2j+6k$

Let $v=\nabla f\times\nabla g$

$=\begin{vmatrix} i & j & k \\ 13 & 13 & -6 \\ 2 & 2 & 6 \end{vmatrix}=i[78+12]-j[78+12]+k[26-26]$

$v=90i-90j$

$\vec{a}=90,\ \vec{b}=-90,\ \vec{c}=0$

The tangent line passes through $(1,1,3)$ and is parallel

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to the vector $\vec{V} = 90i - 90j$, its equation is

$x = x_0 + at$

$x = 1 + 90t$

$y = 1 - 90t$

$z = z_0 + ct$

$= 3 + 0t = \underline{\underline{3}}$

* Find the parametric eqns for the lines tangent to t[he] curve of intersection of the given surfaces at the given point

1) $x + y^2 + 2z = 4$, $x = 1$, $(1, 1, 1)$

→ $f(x, y, z) = x + y^2 + 2z$

$g(x, y, z) = x$

$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} \hat{k}$

$= (1) i + (2y)\hat{j} + 2\hat{k}$

$\nabla f)_{(1,1,1)} = i + 2j + 2\hat{k}$

$\nabla g]_{(1,1,1)} = i$

$v = \nabla f \times \nabla g$

$= \begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 0i - j(-2) + k(-2)$

$= 2j - 2\hat{k}$; $\vec{a} = 0$, $\vec{b} = 2$, $\vec{c} = -2$

$x = 1 + 0t = 1$

$y = 1 + 2t$

$z = 1 - 2t$

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(35)

2) $f(x,y,z) = x^2+y^2$.

$g(x,y,z) = x+z$

$$\nabla f\Big]_{(1,1,3)} = (2x)i + (2y)j + 0\times k$$
$$= 2i + 2j$$

$$\nabla g\Big]_{(1,1,3)} = 1i + 1k$$
$$= \underline{i+k}$$

$$v = \nabla f \times \nabla g$$
$$= \begin{vmatrix} i & j & k \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = i(2-0) - j(2-0) + k(0-2)$$

$$= \underline{2i - 2j - 2k}$$

$\vec{a} = 2$, $\vec{b} = -2$, $\vec{c} = -2$

$x = 1+2t$

$y = 1+(-2t) = 1-2t$

$z = 3-2t$

3) $f(x,y,z) = x^2+2y+2z$

$g(x,y,z) = y$

$$\nabla f\Big]_{(1,1,\frac{1}{2})} = 2x\,i + 2j + 2k$$
$$= 2i + 2j + 2k$$

$$\nabla g\Big]_{(1,1,\frac{1}{2})} = \underline{1j}$$

$$V = \Delta f \times \nabla g = \begin{vmatrix} i & j & k \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{vmatrix} = i(-2) - j(0) + k(2)$$

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$\vec{v} = -2i+2k$

$\vec{a} = -2, \vec{b} = 0, \vec{c} = 2$

$x = 1-2t$

$y = 1+0(t) = 1$

$z = \frac{1}{2} + 2t$

→ 4) $f(x,y,z) = x+y^2+z$

$g(x,y,z) = y$

$\nabla f\Big]_{\frac{1}{2},1,\frac{1}{2}} = 1i + 2yj + 1k$

$= i + \underline{\underline{2j}} + k$

$\nabla g\Big]_{\frac{1}{2},1,\frac{1}{2}} = \underline{\underline{1j}}$

$$\vec{v} = \nabla f \times \nabla g = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

$= i(0-1) - j(0) + k(1)$

$= -i + k$

$\vec{a} = -1, \vec{b} = 0, \vec{c} = 1$

$x = \frac{1}{2} - t$

$y = 1-(0)t = \underline{\underline{1}}$

$z = \frac{1}{2} + t$

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Planes Tangent to a Surface

The eqn of the plane tangent to the surface $z = f(x, y, z)$ at the point $P_0(x_0, y_0, z_0)$ is,

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

* Find the tangent to the surface $z = \ln(x^2+y^2)$ at $(1,0,0)$

→ $f(x, y, z) = \ln(x^2+y^2)$

$$f_x = \frac{1}{x^2+y^2} \times 2x = \frac{2x}{x^2+y^2}$$

$$f_x(x_0, y_0) = \frac{2\times 1}{1^2+0^2} = 2/1 = \underline{\underline{2}}$$

$$f_y = \frac{2y}{x^2+y^2}$$

$$f_y(1,0) = \frac{2\times 0}{1^2+0^2} = \underline{\underline{0}}$$

∴ The eqn is

$$f_x(x_0 y_0)(x - x_0) + f_y(x_0 y_0)(y - y_0) - (z - z_0) = 0$$

$$2(x-1) + 0(y-0) - (z-0) = 0$$

$$2x - 2 - z = 0 \Rightarrow \underline{\underline{2x - z - 2 = 0}}$$

* $z = \ln(x^2+y^2)$ at $(0,1,0)$

→ $f(x, y, z) = \ln(x^2+y^2)$

$$f_x(0,1) = \frac{2x}{x^2+y^2} = \frac{2\times 0}{0^2+1^2} = \underline{\underline{0}}$$

$$f_y(0,1) = \frac{2y}{x^2+y^2} = \frac{2}{0+1} = \underline{\underline{2}}$$

The eqn to the plane tangent

$$0(x-0) + 2(y-1) - (z-0) = 0$$

$$2y - 2 - z = 0 \Rightarrow \underline{\underline{2y - z - 2 = 0}}$$

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* $z = e^{-(x^2+y^2)}$ at $(0,0,1)$

$\rightarrow f(x,y,z) = e^{-(x^2+y^2)}$

$f_x(0,0) = -2e^{-x^2-y^2} \cdot x$

$= -2 \times e \cdot 0 = 0$

$f_y(0,0) = -2e^{-x^2-y^2} \cdot y$

$= -2 \times e^{-1} \times 0 = 0$

The eqn is,

$0(x-0) + 0(y-0) - (z-1) = 0$

$-z+1 = 0$

$\underline{z = 1}$

* $z = \sqrt{y-x}$, $(1,2,1)$

$\rightarrow f_x(1,2) = -\frac{1}{2(y-x)^{1/2}} = -\frac{1}{2(2-1)^{1/2}}$

$= -\frac{1}{2}$

$f_y(1,2) = \frac{1}{2(y-x)^{1/2}} = \frac{1}{2}$

The eqn to the plane tangent is,

$-\frac{1}{2}(x-1) + \frac{1}{2}(y-2) - (z-1) = 0$

$-\frac{x}{2} + \frac{1}{2} + \frac{y}{2} - 1 - z + 1 = 0$

$-\frac{1}{2}(x - 1 - y + 2z) = 0$

$\underline{x - y + 2z - 1 = 0}$

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57) * $z = x\cos y - ye^x$, $(0,0,0)$

→ $f_x(0,0) = -e^x y + \cos(y)$

$= -e^0 \cdot 0 + \cos(0)$

$= 0 + 1 = 1$

$f_y(0,0) = -x\sin(y) - e^x$

$= -0(\sin(0)) - e^0$

$= -1$

The eqn is

$1(x-0) + (-1)(y-0) - (z-0) = 0$

$x - y - z = 0$

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