

## CHAPTER: 1

# VECTOR VALUED FUNCTIONS.

### Vector.

A vector is a quantity that has both magnitude and direction.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2} \Rightarrow \text{Modulus of a vector.}$$

Suppose,  $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$  eq

$$\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$$

Then  $\vec{a} + \vec{b} = (x_1 + x_2)\hat{i} + (y_1 + y_2)\hat{j} + (z_1 + z_2)\hat{k}$

$$k\vec{a} = kx_1\hat{i} + ky_1\hat{j} + kz_1\hat{k}$$

$$\vec{a} - \vec{b} = (x_1 - x_2)\hat{i} + (y_1 - y_2)\hat{j} + (z_1 - z_2)\hat{k}.$$

- If a function given in the form of a vector, then it is, called vector-valued function. And it is written by,

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

• Eg:  $r(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$

$$r(t) = 2t\hat{i} + t\hat{k}$$

### Limit

Consider a vector valued function  $r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

Then  $L$  is said to be the limit of  $r(t)$  at  $t=a$  if,

$$\lim_{t \rightarrow a} r(t) = L$$

• Eg:  $r(t) = t\hat{i} + e^t\hat{j} - \cos t\hat{k}$

Then limit at  $t=0$  is,

$$\lim_{t \rightarrow 0} r(t) = \lim_{t \rightarrow 0} [t\hat{i} + e^t\hat{j} - \cos t\hat{k}]$$

$$= \lim_{t \rightarrow 0} t\hat{i} + \lim_{t \rightarrow 0} e^t\hat{j} - \lim_{t \rightarrow 0} \cos t\hat{k}$$

$$= 0\hat{i} + e^0\hat{j} - \cos 0\hat{k}$$

$$= 0 + \hat{j} - \hat{k}$$

$$= \underline{\underline{\hat{j} - \hat{k}}}$$

### Continuity

$r(t)$  is said to be continuous at  $t=a$

if  $\lim_{t \rightarrow a} r(t) = r(a)$

• eg:  $r(t) = t\hat{i} - \frac{\sin t}{t}\hat{j} + t^2\hat{k}$

then continuity at  $t=0$  is,

$$\begin{aligned} \lim_{t \rightarrow 0} r(t) &= \lim_{t \rightarrow 0} \left[ t\hat{i} - \frac{\sin t}{t}\hat{j} + t^2\hat{k} \right] \\ &= \lim_{t \rightarrow 0} t\hat{i} - \lim_{t \rightarrow 0} \frac{\sin t}{t}\hat{j} + \lim_{t \rightarrow 0} t^2\hat{k} \\ &= 0\hat{i} - \frac{\sin 0}{0}\hat{j} + 0\hat{k} \\ &= \underline{\underline{-\hat{j}}} \end{aligned}$$

over division possible alla  
 $\frac{\sin 0}{0}$  not defined.

$\& r(0) = 0\hat{i} - \frac{\sin 0}{0}\hat{j} + 0\hat{k}$ . which is not defined.  $\therefore \lim_{t \rightarrow 0} r(t) \neq r(0)$

$\therefore$  It is not continuous.

Find the limit of  $r(t) = t^2\hat{i} + (3t-2)\hat{j} + (t^3+1)\hat{k}$  at  $t=0$ .

$\Rightarrow$  So the limit at  $t=0$  is,

$$\begin{aligned} \lim_{t \rightarrow 0} r(t) &= \lim_{t \rightarrow 0} \left[ t^2\hat{i} + (3t-2)\hat{j} + (t^3+1)\hat{k} \right] \\ &= \lim_{t \rightarrow 0} t^2\hat{i} + \lim_{t \rightarrow 0} (3t-2)\hat{j} + \lim_{t \rightarrow 0} (t^3+1)\hat{k} \\ &= 0\hat{i} + (3 \times 0 - 2)\hat{j} + (0+1)\hat{k} \\ &= 0 + (-2)\hat{j} + \hat{k} \\ &= -2\hat{j} + \hat{k} \end{aligned}$$

Q: Find the limit of  $r(t) = \sin t \hat{i} + \cos t \hat{j} + (t+1)\hat{k}$  at  $t = \pi/2$

⇒ limit at  $t = \pi/2$  is,

$$\begin{aligned}\lim_{t \rightarrow \pi/2} r(t) &= \lim_{t \rightarrow \pi/2} [\sin t \hat{i} + \cos t \hat{j} + (t+1)\hat{k}] \\ &= \lim_{t \rightarrow \pi/2} \sin t \hat{i} + \lim_{t \rightarrow \pi/2} \cos t \hat{j} + \lim_{t \rightarrow \pi/2} (t+1) \hat{k} \\ &= \sin \pi/2 \hat{i} + \cos \pi/2 \hat{j} + (\pi/2 + 1) \hat{k} \\ &= 1 \hat{i} + 0 \hat{j} + (\pi/2 + 1) \hat{k} \\ &= \underline{\underline{\hat{i} + (\pi/2 + 1) \hat{k}}}\end{aligned}$$

Q: Check the continuity of  $r(t) = t\hat{i} + a\hat{j} + (a^2 - t^2)\hat{k}$  at  $t = 0$

⇒ The continuity at  $t = 0$ ,

$$\begin{aligned}\lim_{t \rightarrow 0} r(t) &= \lim_{t \rightarrow 0} [t\hat{i} + a\hat{j} + (a^2 - t^2)\hat{k}] \\ &= \lim_{t \rightarrow 0} t\hat{i} + \lim_{t \rightarrow 0} a\hat{j} + \lim_{t \rightarrow 0} (a^2 - t^2)\hat{k} \\ &= 0\hat{i} + a\hat{j} + a^2\hat{k} \\ &= \underline{\underline{a\hat{j} + a^2\hat{k}}}\end{aligned}$$

$$\begin{aligned}\text{now, } r(0) &= 0\hat{i} + a\hat{j} + a^2\hat{k} \\ &= \underline{\underline{a\hat{j} + a^2\hat{k}}}\end{aligned}$$

$$\text{So } \lim_{t \rightarrow 0} r(t) = r(a)$$

ie; it is continuous.

check the continuity of  $r(t) = \cos t \hat{i} + \sin t \hat{j} + t \hat{k}$ , at  $t = \pi/2$

The continuity at  $t = \pi/2$ .

$$\lim_{t \rightarrow \pi/2} r(t) = \lim_{t \rightarrow \pi/2} [\cos t \hat{i} + \sin t \hat{j} + t \hat{k}]$$

$$= \lim_{t \rightarrow \pi/2} \cos t \hat{i} + \lim_{t \rightarrow \pi/2} \sin t \hat{j} + \lim_{t \rightarrow \pi/2} t \hat{k}$$

$$= \cos \pi/2 \hat{i} + \sin \pi/2 \hat{j} + \pi/2 \hat{k}$$

$$= 0 \hat{i} + \hat{j} + \pi/2 \hat{k}$$

$$= \underline{\underline{\hat{j} + \pi/2 \hat{k}}}$$

$$\text{and } r(a) = r(\pi/2)$$

$$r(\pi/2) = \cos \pi/2 \hat{i} + \sin \pi/2 \hat{j} + \pi/2 \hat{k}$$

$$= 0 \hat{i} + \hat{j} + \pi/2 \hat{k}$$

$$= \underline{\underline{\hat{j} + \pi/2 \hat{k}}}$$

$$\text{So } \lim_{t \rightarrow \pi/2} r(t) = r(a)$$

$\therefore$  It is continuous.

## Derivatives

consider a vector-valued function  $r(t) = x(t) \hat{i} +$

$y(t)\hat{j} + z(t)\hat{k}$ . Then the derivative of the vector valued function is  $r'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}$ .

Eg: ①  $r(t) = \sin t \hat{i} + \cos t \hat{j} - t^3 \hat{k}$

$$r'(t) = \cos t \hat{i} - \sin t \hat{j} - 3t^2 \hat{k}$$

②  $r(t) = \sin t \hat{i} + t^3 \hat{j} - e^t \hat{k}$

$$r'(t) = \cos t \hat{i} + 3t^2 \hat{j} - e^t \hat{k}$$

consider two vector valued functions  $r_1(t)$  &  $r_2(t)$ . Then the derivative of their dot product is,

$$\frac{d}{dt} [r_1(t) \cdot r_2(t)] = r_1(t) \cdot \frac{d}{dt} r_2(t) + r_2(t) \cdot \frac{d}{dt} r_1(t)$$

consider two vector valued functions  $r_1(t)$  &  $r_2(t)$ . Then the derivative of their cross product is,

$$\frac{d}{dt} [r_1(t) \times r_2(t)] = r_1(t) \times \frac{d}{dt} r_2(t) + r_2(t) \times \frac{d}{dt} r_1(t)$$

Find the derivative of the following;

$$r(t) = t e^t \hat{i} + \cos t \hat{j}$$

$$r'(t) = \frac{d}{dt} t \cdot e^t \hat{i} + \frac{d}{dt} \cos t \hat{j}$$

applying product rule,

$$\therefore \frac{d}{dt}(t \cdot e^t) = t \cdot \frac{de^t}{dt} + e^t \cdot 1$$

$$= t \cdot e^t + e^t$$

$$\therefore r'(t) = \underline{t \cdot e^t + e^t} \hat{i} - \sin t \hat{j}$$

$$2) \quad r(t) = 2t \hat{i} - 3t^2 \hat{k}$$

$$r'(t) = \underline{2 \hat{i} - 6t \hat{k}}$$

$$3) \quad r(t) = \cos t \hat{i} + e^t \hat{j} + (t \cdot \ln t) \hat{k}$$

$$= -\sin t \hat{i} + e^t \hat{j} + \left[ t \cdot \frac{d \ln t}{dt} + \ln t \cdot \frac{dt}{dt} \right] \hat{k}$$

$$= -\sin t \hat{i} + e^t \hat{j} + \left[ t \cdot \frac{1}{t} + \ln t \cdot 1 \right] \hat{k}$$

$$= \sin t \hat{i} + e^t \hat{j} + [1 + \ln t] \hat{k}$$

$$= \sin t \hat{i} + e^t \hat{j} + \underline{[\ln t + 1] \hat{k}}$$

$$4) \quad r(t) = 2t \hat{i} + t \hat{j} + 3t \hat{k}$$

$$r'(t) = \underline{2 \hat{i} + \hat{j} + 3 \hat{k}}$$

$$\frac{d}{dt} \cos 4t$$

$$= \cos \cdot \frac{d 4t}{dt} + 4t \cdot \frac{d \cos}{dt}$$

$$= -4 \sin$$

$$5) \quad r(t) = \cos 4t \hat{i} + \sin 4t \hat{j} + t \hat{k}$$

$$= -4 \sin 4t \hat{i} + 4 \cos 4t \hat{j} + \hat{k}$$

$$6) \quad r(t) = t \hat{i} + t^2 \hat{j} + t^3 \hat{k}$$

$$= \hat{i} + \underline{2t \hat{j} + 3t^2 \hat{k}}$$

Let  $r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$  be vector valued function. Then the integral of vector function is,

$$\int_a^b r(t) dt = \int_a^b x(t) \cdot \hat{i} + \int_a^b y(t) \hat{j} + \int_a^b z(t) \hat{k}$$

Eg:

$$r(t) = t^3 \hat{i} + 3t^2 \hat{j} + t \hat{k}; \quad 0 \leq t \leq 1$$

$$\Rightarrow \int_0^1 r(t) dt = \int_0^1 [t^3 \hat{i} + 3t^2 \hat{j} + t \hat{k}] dt$$

$$= \int_0^1 t^3 dt \hat{i} + \int_0^1 3t^2 dt \hat{j} + \int_0^1 t dt \hat{k}$$

$$= \left[ \frac{t^4}{4} \hat{i} \right]_0^1 + \left[ \frac{3t^3}{3} \hat{j} \right]_0^1 + \left[ \frac{t^2}{2} \hat{k} \right]_0^1$$

$$= \left[ \frac{t^4}{4} \hat{i} \right]_0^1 + \left[ t^3 \hat{j} \right]_0^1 + \left[ \frac{t^2}{2} \hat{k} \right]_0^1$$

$$= \left[ \frac{1}{4} - 0 \right] \hat{i} + (1^3 - 0) \hat{j} + \left( \frac{1}{2} - 0 \right) \hat{k}$$

$$= \underline{\underline{\frac{1}{4} \hat{i} + \hat{j} + \frac{1}{2} \hat{k}}}$$



## HW

Q: Find the integral of the following?

①  $r(t) = t^2 \hat{i} - 2t \hat{j} + t^4 e^t$ ;  $0 \leq t \leq 2$

$$\Rightarrow \int_0^2 r(t) dt = \int_0^2 (t^2 \hat{i} - 2t \hat{j} + t^4 e^t) dt.$$

$$= \int_0^2 t^2 \hat{i} dt - \int_0^2 2t \hat{j} dt + \int_0^2 t^4 e^t dt.$$

$$= \left[ \frac{t^3}{3} \hat{i} \right]_0^2 - \left[ \frac{2t^2}{2} \hat{j} \right]_0^2 + \left[ \frac{t^5}{5} e^t \right]_0^2$$

$$= \left( \frac{8}{3} - 0 \right) \hat{i} - (4 - 0) \hat{j} + \left( \frac{32}{5} - 0 \right)$$

$$= \frac{8}{3} \hat{i} - 4 \hat{j} + \frac{32}{5}$$

②  $r(t) = \frac{2}{t} \hat{j} + t \hat{k}$ ;  $1 \leq t \leq 3$

$$\Rightarrow \int_1^3 r(t) dt = \int_1^3 \left( \frac{2}{t} \hat{j} + t \hat{k} \right) dt = \left[ 2 \ln t \hat{j} + \frac{t^2}{2} \hat{k} \right]_1^3$$

$$= \left[ 2 \ln 3 \hat{j} + \frac{3^2}{2} \hat{k} \right] - \left[ 2 \ln 1 \hat{j} + \frac{1^2}{2} \hat{k} \right]$$

$$= \left[ 2 \ln 3 \hat{j} + \frac{9}{2} \hat{k} \right] - \left[ 2 \times 0 \hat{j} + \frac{1}{2} \hat{k} \right]$$

$$= 2 \ln 3 \hat{j} + \frac{9}{2} \hat{k} - \left[ 0 \hat{j} + \frac{1}{2} \hat{k} \right] \quad [\because \ln 1 = 0]$$

$$= 2 \ln 3 \hat{j} + \frac{9}{2} \hat{k} - \frac{1}{2} \hat{k}$$

$$= 2 \ln 3 \hat{j} + \left( \frac{9}{2} - \frac{1}{2} \right) \hat{k}$$

$$= 2 \ln 3 \hat{j} + \frac{8}{2} \hat{k} = \underline{\underline{2 \ln 3 \hat{j} + 4 \hat{k}}}$$

$$r(t) = \cos 4t \hat{i} + \sin 4t \hat{j} + t \hat{k}, \quad 0 \leq t \leq 2\pi$$

$$\begin{aligned} \int_0^{2\pi} r(t) dt &= \int_0^{2\pi} (\cos 4t \hat{i} + \sin 4t \hat{j} + t \hat{k}) dt \\ &= \int_0^{2\pi} (\cos 4t \hat{i}) dt + \int_0^{2\pi} (\sin 4t \hat{j}) dt + \int_0^{2\pi} (t \hat{k}) dt \\ &= \left[ \frac{\sin 4t}{4} \hat{i} + \frac{-\cos 4t}{4} \hat{j} + \frac{t^2}{2} \hat{k} \right]_0^{2\pi} \\ &= \left[ \frac{\sin 4(2\pi)}{4} \hat{i} - \frac{\cos 4(2\pi)}{4} \hat{j} + \frac{2\pi^2}{2} \hat{k} \right] - \left[ \frac{\sin 4 \times 0}{4} \hat{i} - \frac{\cos 4 \times 0}{4} \hat{j} + \frac{0}{2} \hat{k} \right] \\ &= \left[ 0 \hat{i} - \frac{1}{4} \hat{j} + \frac{4\pi^2}{2} \hat{k} \right] - \left[ \frac{0}{4} \hat{i} - \frac{1}{4} \hat{j} + 0 \hat{k} \right] \\ &= -\frac{1}{4} \hat{j} + \frac{4\pi^2}{2} \hat{k} + \frac{1}{4} \hat{j} \\ &= \frac{4\pi^2}{2} \hat{k} = \underline{\underline{2\pi^2 \hat{k}}} \end{aligned}$$

$$r(t) = a \cos t \hat{j} + a \sin t \hat{k}, \quad 0 \leq t \leq 2\pi$$

$$\begin{aligned} \int_0^{2\pi} (r(t)) dt &= \int_0^{2\pi} (a \cos t \hat{j} + a \sin t \hat{k}) dt \\ &= \int_0^{2\pi} (a \cos t \hat{j}) dt + \int_0^{2\pi} (a \sin t \hat{k}) dt \\ &= a \int_0^{2\pi} (\cos t \hat{j}) dt + a \int_0^{2\pi} (\sin t \hat{k}) dt \\ &= a \left[ \sin t \right]_0^{2\pi} \hat{j} + a \left[ -\cos t \right]_0^{2\pi} \hat{k} \\ &= a \left[ \sin 2\pi - \sin 0 \right] \hat{j} + a \left[ -\cos 2\pi + \cos 0 \right] \hat{k} \\ &= a \left[ \sin 2\pi - \sin 0 \right] \hat{j} + a \left[ -\cos 2\pi + \cos 0 \right] \hat{k} \\ &= \underline{\underline{0}} \end{aligned}$$

$$r(t) = t\hat{i} + \frac{t^2}{2}\hat{j}, \quad 0 \leq t \leq 2$$

$$\frac{x}{2} = \frac{2 \times 2}{2 \times 2}$$

$$\int_0^2 r(t) dt = \int_0^2 (t\hat{i}) dt + \int_0^2 \left(\frac{t^2}{2}\hat{j}\right) dt$$

$$= \left[\frac{t^2}{2}\right]_0^2 \hat{i} + \left[\frac{t^3}{2 \times 3}\right]_0^2 \hat{j}$$

$$= \left[\frac{4}{2} - 0\right] \hat{i} + \left[\frac{8}{6}\right] \hat{j}$$

$$= 2\hat{i} + \frac{4}{3}\hat{j} \quad \checkmark$$

$$\frac{\frac{2^2}{2}}{2} \times \frac{2}{1}$$

$$\frac{\pi^2}{2} \times 1 - \frac{\pi^2}{2}$$

$$\frac{\pi^2}{4} / 2 \rightarrow 180$$

$$r(t) = \cos t \hat{i} + \sin t \hat{j} + t \hat{k}, \quad 0 \leq t \leq \frac{\pi}{2}$$

$$\int_0^{\pi/2} r(t) dt = \int_0^{\pi/2} (\cos t) \hat{i} dt + \int_0^{\pi/2} (\sin t) \hat{j} dt + \int_0^{\pi/2} t \hat{k} dt$$

$$= \left[\sin t \hat{i}\right]_0^{\pi/2} + \left[-\cos t \hat{j}\right]_0^{\pi/2} + \left[\frac{t^2}{2} \hat{k}\right]_0^{\pi/2}$$

$$= \left[\sin \frac{\pi}{2} - \sin 0\right] \hat{i} + \left[-\cos \frac{\pi}{2} + \cos 0\right] \hat{j} + \left[\frac{\pi^2}{2} - 0\right] \hat{k}$$

$$= \left[\frac{\pi^2}{2} - 0\right] \hat{k} + \left[\sin \frac{\pi}{2} - \cos \frac{\pi}{2}\right] \hat{i} + \left[\frac{\pi^2}{2}\right] \hat{k} - \left[\sin 0 \hat{i} - \cos 0 \hat{j} + 0 \frac{\pi^2}{2} \hat{k}\right]$$

$$= \left[\frac{\pi^2}{4} \times 2 - 0\right] \hat{k} + \left[\cos 0 + 0 \frac{\pi^2}{2} \hat{k}\right]$$

$$= \left[\frac{\pi^2}{4} \times 2 - 0\right] \hat{k} = \left[\frac{\pi^2}{2}\right] \hat{k}$$

$$= \frac{\pi^2}{4} \hat{k}$$

$$= \hat{i} + \frac{\pi^2}{8} \hat{k} + \hat{j}$$

$$= \hat{i} + \hat{j} + \frac{\pi^2}{8} \hat{k}$$

$$\underline{\underline{\hat{i} + \hat{j} + \frac{\pi^2}{8} \hat{k}}}$$

Consider a vector valued function  $r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$  and its derivative  $r'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}$ .

Then the length of a space curve is,

$$L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} dt$$

Eg:

Find the length of the curve  $r(t) = \langle 6\cos(2t), 6\sin(2t), 5t \rangle$  for  $t \in [0, \pi]$ ?

$$r(t) = 6\cos(2t)\hat{i} + 6\sin(2t)\hat{j} + 5t\hat{k}$$

$$r'(t) = 6 \times -\sin(2t) \times 2\hat{i} + 6 \times \cos(2t) \times 2\hat{j} + 5\hat{k}$$

$$r'(t) = -12\sin(2t)\hat{i} + 12\cos(2t)\hat{j} + 5\hat{k}$$

here, when  $a = 0, b = \pi$ .

$$\therefore L = \int_0^{\pi} \sqrt{(-12\sin(2t))^2 + (12\cos(2t))^2 + (5)^2} dt$$

$$L = \int_0^{\pi} \sqrt{144\sin^2(2t) + 144\cos^2(2t) + 25} dt$$

$$= \int_0^{\pi} \sqrt{144[\sin^2(2t) + \cos^2(2t)] + 25} dt$$

$$= \int_0^{\pi} \sqrt{144 \times 1 + 25} dt$$

$$= \int_0^{\pi} \sqrt{169} \, dt$$

$$= \int_0^{\pi} 13 \, dt$$

$$= 13t \Big|_0^{\pi} = \underline{\underline{13\pi}}$$

Hw (Find the length of the following)

$$\textcircled{1} \, r(t) = (4 \cos t \hat{i} + 4 \sin t \hat{j} + 3t \hat{k}) \quad [0, 1]$$

$$\therefore r(t) = 4 \cos t \hat{i} + 4 \sin t \hat{j} + 3t \hat{k}$$

$$r'(t) = -4 \sin t \hat{i} + 4 \cos t \hat{j} + 3 \hat{k}$$

$$= \underline{\underline{-4 \sin t \hat{i} + 4 \cos t \hat{j} + 3 \hat{k}}}$$

$$a = 0 \quad \& \quad b = 1$$

$$\therefore L = \int_0^1 \sqrt{(-4 \sin t)^2 + (4 \cos t)^2 + 3^2} \, dt$$

$$= \int_0^1 \sqrt{16 \sin^2 t + 16 \cos^2 t + 9} \, dt$$

$$= \int_0^1 \sqrt{16(\sin^2 t + \cos^2 t) + 9} \, dt$$

$$= \int_0^1 \sqrt{16 \times 1 + 9} \, dt$$

$$= \int_0^1 \sqrt{25} \, dt$$

$$= \int_0^1 5 \, dt$$

$$= 5t \Big|_0^1 = 5 \times 1 - 0 = \underline{\underline{5}}$$

$$\textcircled{2} \quad r(t) = t\hat{i} + \cosh t\hat{j}$$

$$r'(t) = \hat{i} + \sinh t\hat{j}$$

$$x'(t) = 1, \quad y'(t) = \sinh t$$

$$a=0, \quad b=1$$

$$L = \int_0^1 \sqrt{1^2 + (\sinh t)^2} dt$$

$$= \int_0^1 \sqrt{1 + \sinh^2 t} dt$$

$$= \int_0^1 \sqrt{\cosh 2t} dt = \int_0^1 \cosh t dt$$

$$= \left[ \sinh t \right]_0^1 = \sinh 1 - \sinh 0$$

$[\because \sinh 0 = 0]$

$$= \underline{\underline{\sinh 1}}$$

$$r(t) = \cos t\hat{i} + \sin t\hat{j} + t\hat{k}; \quad t \in [0, 2\pi]$$

$$r'(t) = \underline{\underline{-\sin t\hat{i} + \cos t\hat{j} + \hat{k}}}$$

$$a=0 \quad b=2\pi$$

$$\therefore L = \int_0^{2\pi} \sqrt{-\sin t\hat{i} + \cos t\hat{j} + \hat{k}} dt$$

$$= \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} dt$$

$$= \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1} dt$$

$$= \int_0^{2\pi} \sqrt{1+1} dt$$

$$= \int_0^{2\pi} \sqrt{2} dt = \underline{\underline{\sqrt{2} \cdot 2\pi}}$$

$$3) \gamma(t) = t\mathbf{i} + \sin 2\pi t \mathbf{j} + \cos 2\pi t \mathbf{k}, \quad t \in (0, 1)$$

$$\gamma'(t) = \mathbf{i} + \cos 2\pi t \mathbf{j} - \sin 2\pi t \mathbf{k}$$

$$= \mathbf{i} + \cos 2\pi t \mathbf{j} \times 2\pi \mathbf{j} + -\sin 2\pi t \times 2\pi \mathbf{k}$$

$$\gamma'(t) = \underline{\underline{\mathbf{i} + 2\pi \cos 2\pi t \mathbf{j} - 2\pi \sin 2\pi t \mathbf{k}}}$$

$$x'(t) = 1, \quad y'(t) = 2\pi \cos 2\pi t, \quad z'(t) = -2\pi \sin 2\pi t$$

$$a = 0, \quad b = 1$$

$$\therefore L = \int_0^1 \sqrt{(1)^2 + (2\pi \cos 2\pi t)^2 + (-2\pi \sin 2\pi t)^2} dt$$

$$= \int_0^1 \sqrt{1 + 4\pi^2 \cos^2 2\pi t + 4\pi^2 \sin^2 2\pi t} dt$$

$$= \int_0^1 \sqrt{1 + 4\pi^2 (\cos^2 2\pi t + \sin^2 2\pi t)} dt$$

$$= \int_0^1 \sqrt{1 + 4\pi^2 \times 1} dt$$

$$= \int_0^1 \sqrt{1 + 4\pi^2} dt$$

$$= \sqrt{1 + 4\pi^2} \int_0^1 dt$$

$$= \sqrt{1 + 4\pi^2} \times [t]_0^1$$

$$= \sqrt{1 + 4\pi^2} \times 1$$

$$= \underline{\underline{\sqrt{1 + 4\pi^2}}}$$

$$5) \gamma(t) = 2\cos t \mathbf{i} + 2\sin t \mathbf{j} + \sqrt{5}t \mathbf{k}, \quad 0 \leq t \leq \pi$$

$$\gamma'(t) = \underline{\underline{-2\sin t \mathbf{i} + 2\cos t \mathbf{j} + \sqrt{5} \mathbf{k}}}$$

$$x'(t) = -2 \sin t, \quad y'(t) = 2 \cos t, \quad z'(t) = \sqrt{5}$$

$$a = 0, \quad b = \pi$$

$$\therefore L = \int_0^{\pi} \sqrt{(-2 \sin t)^2 + (2 \cos t)^2 + (\sqrt{5})^2} dt$$

$$= \int_0^{\pi} \sqrt{(-2)^2 \sin^2 t + 2^2 \cos^2 t + 5} dt$$

$$= \int_0^{\pi} \sqrt{4 \sin^2 t + 4 \cos^2 t + 5} dt$$

$$= \int_0^{\pi} \sqrt{4 [\sin^2 t + \cos^2 t] + 5} dt$$

$$= \int_0^{\pi} \sqrt{4 \times 1 + 5} dt$$

$$= \int_0^{\pi} \sqrt{9} dt$$

$$= \int_0^{\pi} 3 dt$$

$$= 3\pi - 0 = \underline{\underline{3\pi}}$$

$$r(t) = ti + \left(\frac{2}{3}\right) t^{3/2} k, \quad 0 \leq t \leq 8.$$

$$r'(t) = i + \frac{2}{3} \times \frac{3}{2} t^{1/2} k$$

$$= \underline{\underline{i + t^{1/2} k}}$$

$$x'(t) = 1, \quad y'(t) = t^{1/2}$$

$$a = 0, \quad b = 8$$

$$L = \int_0^8 \sqrt{(1)^2 + (t^{1/2})^2} dt$$

$$= \int_0^8 \sqrt{1+t} dt.$$



$$= \int_0^8 (1+t)^{3/2} dt.$$

$$= \left[ \frac{(1+t)^{5/2}}{5/2} \right]_0^8 \quad \left[ \because \int x^n = \frac{x^{n+1}}{n+1} \right]$$

$$= \frac{2}{5} (1+t)^{5/2} \Big|_0^8$$

$$= \frac{2}{5} \left[ (1+8)^{5/2} - (1+0)^{5/2} \right]$$

$$= \frac{2}{5} \left[ 9^{5/2} - 1^{5/2} \right]$$

$$= \frac{2}{5} [27 - 1] = \frac{2}{5} \times 26 = \underline{\underline{\frac{52}{5}}}$$

7)  $r(t) = (2+t)\mathbf{i} - (t+1)\mathbf{j} + t\mathbf{k} ; 0 \leq t \leq 3$ .

$$r'(t) = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$x'(t) = 1, y'(t) = -1, z'(t) = 1$$

$$a=0, b=3$$

$$\therefore L = \int_0^3 \sqrt{1^2 + (-1)^2 + 1^2} dt$$

$$= \int_0^3 \sqrt{1+1+1} dt$$

$$= \int_0^3 \sqrt{3} dt$$

$$= \left[ \sqrt{3}t \right]_0^3 \Rightarrow \left[ \sqrt{3} \times 3 - \sqrt{3} \times 0 \right] = \underline{\underline{3\sqrt{3}}}$$

8)  $r(t) = t \cos t \mathbf{i} + t \sin t \mathbf{j} + \left[ \frac{2\sqrt{2}}{3} \right]^{3/2} t \mathbf{k}$

$$r'(t) = [t \times -\sin t + \cos t] \mathbf{i} + [t \cos t + \sin t] \mathbf{j} + \left[ \frac{2\sqrt{2}}{3} \times \frac{3}{2} t \right] \mathbf{k}$$

$$= [-t \sin t + \cos t] \mathbf{i} + [t \cos t + \sin t] \mathbf{j} + \sqrt{2} t^{1/2} \mathbf{k}$$

$$x'(t) = -t \sin t + \cos t, \quad y'(t) = t \cos t + \sin t, \quad z'(t) = \sqrt{2} t^{1/2}$$

$$a=0, \quad b=\pi.$$

$$L = \int_0^{\pi} \sqrt{(-t \sin t + \cos t)^2 + (t \cos t + \sin t)^2 + (\sqrt{2} t^{1/2})^2} dt$$

$$= \int_0^{\pi} \sqrt{[t^2 \sin^2 t + \cos^2 t - 2t \sin t \cos t] + t^2 \cos^2 t + \sin^2 t + 2t \cos t \sin t + 2t} dt$$

$$= \int_0^{\pi} \sqrt{t^2 \sin^2 t + \cos^2 t + t^2 \cos^2 t + \sin^2 t + 2t} dt$$

$$= \int_0^{\pi} \sqrt{t^2 \sin^2 t + t^2 \cos^2 t + \cos^2 t + \sin^2 t + 2t} dt$$

$$= \int_0^{\pi} \sqrt{t^2 [\sin^2 t + \cos^2 t] + [\cos^2 t + \sin^2 t] + 2t} dt$$

$$= \int_0^{\pi} \sqrt{t^2 \cdot 1 + 1 + 2t} dt = \int_0^{\pi} (t^2 + 2t + 1)^{1/2} dt$$

$$= \left[ \frac{(t^2 + 2t + 1)^{3/2}}{3/2} \right]_0^{\pi} = \left[ \frac{(t^2 + 2t + 1)^{3/2}}{3/2} \times \frac{1}{2t + 2} \right]_0^{\pi}$$

$$= \frac{2}{3} \left[ (t^2 + 2t + 1)^{3/2} \times \frac{1}{2t + 2} \right]_0^{\pi}$$

$$= \left[ \frac{2}{3} \frac{[\pi^2 + 2\pi + 1]}{2\pi + 2} \right] - \left[ \frac{2}{3} \frac{[0^2 + 2 \times 0 + 1]}{2 \times 0 + 2} \right]$$

$$= \frac{2}{3} \frac{[\pi^2 + 2\pi + 1]}{2\pi + 2} - \frac{2}{3} \left[ \frac{1}{2} \right]$$

$$= \frac{2}{3} \times \frac{\pi^2 + 2\pi + 1}{2\pi + 2} - \frac{2/3}{2}$$

$$= \frac{2}{3} \left[ \frac{2\pi^2 + 2\pi}{4\pi + 4} \right]$$

$$= \frac{2}{3} \left[ \frac{2(\pi^2 + \pi)}{4(\pi + 1)} \right]$$

$$= \frac{1}{3} \left[ \frac{\pi^2 + \pi}{\pi + 1} \right]$$

9)  $r(t) = \cos^3 t \mathbf{j} + \sin^3 t \mathbf{k} ; 0 \leq t \leq \pi/2.$

$$r'(t) = 3\cos^2 t \cdot (-\sin t) \mathbf{j} + 3\sin^2 t \cdot \cos t \mathbf{k}.$$

$$r'(t) = -3\cos^2 t \cdot \sin t \mathbf{j} + 3\sin^2 t \cdot \cos t \mathbf{k}$$

$$\therefore x'(t) = -3\cos^2 t \sin t, \quad y'(t) = 3\sin^2 t \cos t.$$

$$a = 0, \quad b = \pi/2$$

$$\therefore L = \int_0^{\pi/2} \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2} dt$$

$$= \int_0^{\pi/2} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} dt.$$

$$= \int_0^{\pi/2} \sqrt{9[\cos^4 t \sin^2 t + \sin^4 t \cos^2 t]} dt$$

$$= \int_0^{\pi/2} \sqrt{9\cos^4 t \sin^2 t [\cos^2 t + \sin^2 t]} dt$$

$$= \int_0^{\pi/2} \sqrt{9\cos^2 t \sin^2 t \times 1} dt$$

$$= \int_0^{\pi/2} \sqrt{9\cos^2 t \sin^2 t} dt$$

$$= \int_0^{\pi/2} 3\cos t \sin t dt.$$

$$= 3 \left[ \cos t \sin t \right]_0^{\pi/2}$$

$$= 3 \left[ (\cos \pi/2 \sin \pi/2) - (\cos 0 \sin 0) \right]$$

$$= 3 \left[ (0 \times 1) - (1 \times 0) \right]$$

$$= 3 \times 0 = 0$$

$$= 3 \int_0^{\pi/2} \cos t \sin t \, dt$$

$$= 3 \left[ \int_0^1 u \, du = \frac{3 \times u^2}{2} \right]_0^1$$

$$= 3 \times \left( \frac{1-0}{2} \right) = \frac{3}{2} //$$

$$\sin t = u$$

$$\cos t \, dt = du$$

$$u = \sin 0 = 0$$

$$u = \sin \pi/2 = 1$$

$$r(t) = e^t \cos t \hat{i} + e^t \sin t \hat{j} \quad 0 \leq t \leq 1$$

$$r'(t) = \cos t \cdot e^t \hat{i} + \sin t \cdot e^t \hat{j}$$

$$r'(t) = [\cos t \cdot e^t + e^t \cdot (-\sin t)] \hat{i} + [\sin t \cdot e^t + e^t \cdot \cos t] \hat{j} \quad *$$

$$r'(t) = [e^t \cos t - \sin t e^t] \hat{i} + [e^t \sin t + e^t \cos t] \hat{j}$$

$$x'(t) = e^t \cos t - \sin t e^t, \quad y'(t) = e^t \sin t + e^t \cos t$$

$$a=0, \quad b=1$$

$$\therefore L = \int_0^1 \sqrt{[e^t \cos t - e^t \sin t]^2 + [e^t \sin t + e^t \cos t]^2} \, dt$$

$$= \int_0^1 \sqrt{[e^{2t} \cos^2 t + e^{2t} \sin^2 t - 2e^{2t} \cos t \sin t] + [e^{2t} \sin^2 t$$

$$+ e^{2t} \cos^2 t + 2e^{2t} \sin t \cos t] \, dt$$

Q:

$$\begin{aligned}
&= \int_0^1 \sqrt{e^{2t} [\cos^2 t + \sin^2 t] + e^{2t} [\sin^2 t + \cos^2 t]} dt \\
&= \int_0^1 \sqrt{e^{2t} \times 1 + e^{2t} \times 1} dt \\
&= \int_0^1 \sqrt{2} \sqrt{e^{2t} + e^{2t}} dt \\
&= \int_0^1 \sqrt{2e^{2t}} dt \\
&= \int_0^1 \sqrt{2} \times \sqrt{e^{2t}} dt \\
&= \int_0^1 \sqrt{2} \times (e^{2t})^{1/2} dt = \sqrt{2} \int_0^1 e^{2t \times 1/2} dt \\
&= \sqrt{2} \int_0^1 e^t dt = \\
&= \sqrt{2} [e^t]_0^1 \\
&= \sqrt{2} [e^1 - e^0] = \underline{\underline{\sqrt{2} [e - 1]}}
\end{aligned}$$

### \* Arc length as a Parameter.

If a curve  $\{r(t)\}$  is already given in terms, of the parameter  $t$ , the arc length parameter with base point  $P(t_0)$  is given by,

$$s(t) = \int_{t_0}^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2} du$$

eg: -

Q: Find the arc length parameterization of

the helix,  $r(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$ ,  $0 \leq t$

$$\Rightarrow r(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$$

$$r'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}$$

$$r'(u) = -\sin u \mathbf{i} + \cos u \mathbf{j} + \mathbf{k}.$$

Here  $t_0 = 0$ .

$$\therefore S(t) = \int_0^t \sqrt{(-\sin u)^2 + (\cos u)^2 + (1)^2} du$$

$$S = \int_0^t \sqrt{\sin^2 u + \cos^2 u + 1} du$$

$$S = \int_0^t \sqrt{1+1} du$$

$$= \int_0^t \sqrt{2} du$$

$$= \sqrt{2} \int_0^t du$$

$$= \sqrt{2} [u]_0^t \Rightarrow \sqrt{2} [t-0]$$

$$S = \underline{\underline{\sqrt{2} t}}$$

$$\text{So, } S = \sqrt{2} t$$

$$\therefore t = \frac{S}{\sqrt{2}}$$

$$\therefore r(t) = r\left(\frac{S}{\sqrt{2}}\right) =$$

$$= \underline{\underline{\cos\left(\frac{S}{\sqrt{2}}\right) \mathbf{i} + \sin\left(\frac{S}{\sqrt{2}}\right) \mathbf{j} + \left(\frac{S}{\sqrt{2}}\right) \mathbf{k}}}$$

This is the <sup>arc</sup> length ~~arc~~ parameterization of the given curve.

Q:  $r(t) = 2 \cos t \hat{i} + 2 \sin t \hat{j}$ ,  $0 \leq t \leq 2\pi$ .

$r'(t) = -2 \sin t \hat{i} + 2 \cos t \hat{j}$

$r'(u) = -2 \sin u \hat{i} + 2 \cos u \hat{j}$

Here,  $t_0 = 0$

$\therefore S(t) = \int_0^t \sqrt{(-2 \sin u)^2 + (2 \cos u)^2} du$

$S = \int_0^t \sqrt{4 \sin^2 u + 4 \cos^2 u} du$

$= \int_0^t \sqrt{4 (\sin^2 u + \cos^2 u)} du$

$= \int_0^t \sqrt{4 \times 1} du$

$= \int_0^t 2 du$

$= 2 \int_0^t du$

$= 2 [u]_0^t \Rightarrow 2 [t - 0]$

$S = 2t$

So  $S = 2t$

$t = \frac{S}{2}$

$\therefore r(t) = r\left(\frac{S}{2}\right)$

$= 2 \cos\left(\frac{S}{2}\right) \hat{i} + 2 \sin\left(\frac{S}{2}\right) \hat{j}$

Q:  $r(t) = 4 \cos t \hat{i} + 4 \sin t \hat{j} + 3t \hat{k}$ ,  $0 \leq t \leq \frac{\pi}{2}$

$r'(t) = -4 \sin t \hat{i} + 4 \cos t \hat{j} + 3 \hat{k}$

$r'(u) = -4 \sin u \hat{i} + 4 \cos u \hat{j} + 3 \hat{k}$

Here,  $t_0 = 0$

$$\therefore s(t) = \int_0^t \sqrt{(-4\sin u)^2 + (4\cos^2 u)^2 + 3^2} du$$

$$s = \int_0^t \sqrt{16\sin^2 u + 16\cos^2 u + 9} du$$

$$= \int_0^t \sqrt{16(\sin^2 u + \cos^2 u) + 9} du$$

$$= \int_0^t \sqrt{16 \times 1 + 9} du$$

$$= \int_0^t \sqrt{25} du$$

$$= 5 \int_0^t du \Rightarrow 5 [u]_0^t$$

$$= 5 [t - 0]$$

$$s = \underline{\underline{5t}}$$

So,  $s = 5t$ .

$$t = \underline{\underline{\frac{s}{5}}}$$

$$\therefore \vec{r}(t) = \vec{r}\left(\frac{s}{5}\right) = 4\cos\left(\frac{s}{5}\right)\mathbf{i} + 4\sin\left(\frac{s}{5}\right)\mathbf{j} + 3\left(\frac{s}{5}\right)\mathbf{k}$$

$$\vec{r}(t) = e^t \cos t \mathbf{i} + e^t \sin t \mathbf{j} + e^t \mathbf{k}$$



$$Q: \quad r(t) = e^t \cos t \hat{i} + e^t \sin t \hat{j}$$

$$r'(t) = (\cos t \times e^t + e^t \times -\sin t) \hat{i} + (\sin t \times e^t + e^t \times \cos t) \hat{j}$$

$$r'(t) = (e^t \cos t - e^t \sin t) \hat{i} + (e^t \sin t + e^t \cos t) \hat{j}$$

$$r'(u) = (e^u \cos u - e^u \sin u) \hat{i} + (e^u \sin u + e^u \cos u) \hat{j}$$

$$t_0 = 0$$

$$s(t) = \int_0^t \sqrt{(e^u \cos u - e^u \sin u)^2 + (e^u \sin u + e^u \cos u)^2} du$$

$$= \int_0^t \sqrt{[(e^u \cos u)^2 + (e^u \sin u)^2 - 2e^u \cos u \times e^u \sin u] + [(e^u \sin u)^2 + (e^u \cos u)^2 + 2e^u \sin u \times e^u \cos u]} du$$

$$= \int_0^t \sqrt{e^{2u} \cos^2 u + e^{2u} \sin^2 u + e^{2u} \sin^2 u + e^{2u} \cos^2 u} du$$

$$= \int_0^t \sqrt{e^{2u} [\cos^2 u + \sin^2 u] + e^{2u} [\sin^2 u + \cos^2 u]} du$$

$$= \int_0^t \sqrt{e^{2u} \times 1 + e^{2u} \times 1} du$$

$$= \int_0^t \sqrt{2e^{2u}} du$$

$$= \int_0^t \sqrt{2} \cdot \sqrt{e^{2u}} du$$

$$= \sqrt{2} \int_0^t \sqrt{e^{2u}} du$$

$$= \sqrt{2} \times \int_0^t (e^{2u})^{1/2} du$$

$$= \sqrt{2} \int_0^t e^{2u \times 1/2} du$$

$$= \sqrt{2} \int_0^t e^{u} du$$

$$= \sqrt{2} \times e^u \Big|_0^t$$

$$= \sqrt{2} [e^t - e^0]$$

$$S = \sqrt{2} [e^t - 1]$$

∴

Q:  $r(t) = e^t \cdot \cos t \, i + e^t \sin t \, j + e^t \, k$ ,  $-\ln 4 \leq t \leq 0$

$$r'(t) = (\cos t \times e^t + e^t \times -\sin t) i + (\sin t \times e^t + e^t \cdot \cos t) j + e^t k$$

$$r'(t) = (e^t \cos t - e^t \sin t) i + (e^t \sin t + e^t \cos t) j + e^t k$$

P:  $r(u) = (e^u \cos u - e^u \sin u) i + (e^u \sin u + e^u \cos u) j + e^u k$

$$t_0 = -\ln 4$$

$$\therefore S(t) = \int_{-\ln 4}^0 \sqrt{(e^u \cos u - e^u \sin u)^2 + (e^u \sin u + e^u \cos u)^2 + (e^u)^2} du$$

$$= \int_{-\ln 4}^0 \sqrt{[(e^u \cos u)^2 + (e^u \sin u)^2 - 2e^u \cos u \times e^u \sin u] +$$

$$[(e^u \sin u)^2 + (e^u \cos u)^2 + 2e^u \sin u \times e^u \cos u] + (e^u)^2} du$$

$$= \int_{-\ln 4}^0 \sqrt{[e^{2u} \cos^2 u + e^{2u} \sin^2 u] + [e^{2u} \sin^2 u + e^{2u} \cos^2 u] + e^{2u}} du$$

$$= \int_{-\ln 4}^0 \sqrt{e^{2u} (\cos^2 u + \sin^2 u) + e^{2u} (\sin^2 u + \cos^2 u) + e^{2u}} du$$

$$\begin{aligned}
&= \int_{-\ln 4}^t \sqrt{e^{2u} \times 1 + e^{2u} \times 1 + e^{2u}} \, du \\
&= \int_{-\ln 4}^t \sqrt{e^{2u} + e^{2u} + e^{2u}} \, du \\
&= \int_{-\ln 4}^t \sqrt{3e^{2u}} \, du \\
&= \sqrt{3} \cdot \int_{-\ln 4}^t \sqrt{e^{2u}} \, du \\
&= \sqrt{3} \int_{-\ln 4}^t (e^{2u})^{1/2} \, du \\
&= \sqrt{3} \int_{-\ln 4}^t e^{2u \times 1/2} \, du \\
&= \sqrt{3} \int_{-\ln 4}^t e^u \, du \\
&= \sqrt{3} \cdot [e^u]_{-\ln 4}^t \\
&= \sqrt{3} \cdot [e^t - e^{-\ln 4}] \\
&= \underline{\underline{\sqrt{3} \cdot [e^t - e^{-\ln 4}]}}
\end{aligned}$$

Q. Find the length of the circular helix.

$$r(t) = a \cos t \, i + a \sin t \, j + ct \, k$$

from  $(a, 0, 0)$  to  $(a, 0, 2\pi c)$  ?

$$\Rightarrow L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt \rightarrow \textcircled{1}$$

$$r(t) = a \cos t \, i + a \sin t \, j + ct \, k$$

$$r'(t) = -a \sin t \mathbf{i} + a \cos t \mathbf{j} + c \mathbf{k}$$

$$x = a \cos t, \quad y = a \sin t, \quad z = ct$$

$$x'(t) = -a \sin t, \quad y'(t) = a \cos t, \quad z'(t) = c$$

Now consider the point  $(a, 0, 0)$ .

$$x = a \cos t \Rightarrow a = a \cos t \Rightarrow \cos t = 1 \Rightarrow t = \cos^{-1}(1) = 0$$

$$y = a \sin t \Rightarrow 0 = a \sin t \Rightarrow \sin t = 0 \Rightarrow t = \sin^{-1}(0) = 0$$

$$z = ct \Rightarrow 0 = ct \Rightarrow t = \underline{\underline{0}}$$

$$\therefore \underline{\underline{a = 0}}$$

Consider  $(a, 0, 2\pi c)$

$$x = a \cos t \Rightarrow a = a \cos t \Rightarrow 1 = \cos t \Rightarrow t = \cos^{-1}(1) = 0$$

$$y = a \sin t \Rightarrow 0 = a \sin t \Rightarrow 0 = \sin t \Rightarrow t = \sin^{-1}(0) = 0$$

$$z = ct \Rightarrow 2\pi c = ct \Rightarrow \underline{\underline{2\pi = t}}$$

$$\therefore \underline{\underline{b = 2\pi}}$$

$$\therefore L = \int_0^{2\pi} \sqrt{(-a \sin t)^2 + (a \cos t)^2 + c^2} dt$$

$$= \int_0^{2\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + c^2} dt$$

$$= \int_0^{2\pi} \sqrt{a^2 (\sin^2 t + \cos^2 t) + c^2} dt$$

$$= \int_0^{2\pi} \sqrt{a^2 + c^2} dt$$

$$= \sqrt{a^2 + c^2} \int_0^{2\pi} dt$$

$$= \left[ \sqrt{a^2 + c^2} \times t \right]_0^{2\pi}$$

$$= (2\pi - 0) \sqrt{a^2 + c^2}$$

$$\underline{\underline{L = 2\pi \sqrt{a^2 + c^2}}}$$

Q: Find the length of the semi cubical parabola.

$r(t) = ti + t^{3/2}j$  from  $(0,0,0)$  to  $(4,8,0)$  ?

$$\Rightarrow L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt$$

$$\therefore r(t) = ti + t^{3/2}j$$

$$r'(t) = i + \frac{3}{2}t^{1/2}j$$

$$x = t \quad y = t^{3/2}$$

$$x' = 1 \quad y' = \frac{3}{2}t^{1/2}$$

Now consider the point,  $(0,0,0)$

$$x = t \Rightarrow 0 = t \Rightarrow \underline{\underline{t = 0}}$$

$$y = t^{3/2} \Rightarrow 0 = t^{3/2} \Rightarrow \underline{\underline{t = 0}}$$

$$\therefore \underline{\underline{a = 0}}$$

Consider  $(4,8,0)$

$$x = t \Rightarrow 4 = t \Rightarrow \underline{\underline{t = 4}}$$

$$y = t^{3/2} \Rightarrow 8 = t^{3/2} \Rightarrow \underline{\underline{t = 8}} \Rightarrow (t^{3/2})^{2/3} = 8^{2/3} t^{1/3} = 8^{2/3} \Rightarrow \sqrt[3]{8^2} = 4$$

$$\therefore \underline{\underline{b = 4}}$$

$$= \underline{\underline{t = 4}}$$

$$\therefore L = \int_0^4 \sqrt{1^2 + \left[\frac{3}{2}t\right]^2} dt$$

$$= \int_0^4 \sqrt{1 + \frac{9}{4}(t^2)^2} dt = \int_0^4 \sqrt{1 + \frac{9}{4}t} dt$$

$$= \int_0^4 \left(1 + \frac{9t}{4}\right)^{1/2} dt$$

$$= \left[ \frac{\left(1 + \frac{9t}{4}\right)^{3/2}}{3/2} \right]_0^4 = \frac{2}{3} \times \left[ \frac{\left(1 + \frac{9t}{4}\right)^{3/2}}{\frac{9}{4}} \right]_0^4$$

$$= \frac{2}{3} \times \frac{4}{9} \times \left[ 1 \times \frac{9t}{4} \right]^{3/2} \Big|_0^4$$

$$= \frac{8}{27} \left[ \left(1 + \frac{9 \times 4}{4}\right)^{3/2} - \left(1 + \frac{9 \times 0}{4}\right)^{3/2} \right]$$

$$= \frac{8}{27} \left[ \left(1 + \frac{36}{4}\right)^{3/2} - (1+0)^{3/2} \right]$$

$$= \frac{8}{27} \left[ (10)^{3/2} - (1)^{3/2} \right]$$

$$= \frac{8}{27} \left[ (10)^{3/2} - 1 \right]$$

Q: Find the length of the curve  $x=t, y=2t+5, z=3t$  from  $(0, 5, 0)$  to  $(1, 7, 3)$

$$\Rightarrow r(t) = t\hat{i} + (2t+5)\hat{j} + 3t\hat{k}$$

$$r'(t) = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$x = t$$

$$y = 2t + 5$$

$$z = 3t$$

$$x'(t) = 1$$

$$y'(t) = 2$$

$$z'(t) = 3$$

Now consider the point,  $(0, 5, 0)$

$$x = t \Rightarrow 0 = t \Rightarrow \underline{t = 0}$$

$$y = 2 \Rightarrow 5 = 2 \Rightarrow t = 0 //$$

$$z = 3 \Rightarrow 0 = 3 = \underline{t = 0}$$

$$\therefore \underline{a = 0}$$

Consider  $(1, 7, 3)$

$$x = t \Rightarrow 1 = t \Rightarrow t = 1$$

$$y = 2t + 5 \Rightarrow 7 = 2t + 5 \Rightarrow 2t + 5 = 7 \Rightarrow 2t = 7 - 5$$

$$y = 2 \Rightarrow 7 = 2 \Rightarrow t = 0$$

$$z = 3t \Rightarrow 3 = 3t \Rightarrow \underline{t = 1}$$

$$\therefore \underline{b = 1}$$

$$\therefore L = \int_0^1 \sqrt{(1)^2 + (2)^2 + (3)^2} dt$$

$$= \int_0^1 \sqrt{1 + 4 + 9} dt$$

$$= \int_0^1 \sqrt{14} dt$$

$$= \sqrt{14} \int_0^1 dt$$

$$= \sqrt{14} [1 - 0]$$

$$= \sqrt{14} \times 1 = \underline{\underline{\sqrt{14}}}$$

Q: Find the length of the curve  $x = \sin t - t \cos t$   
 $y = \cos t + t \sin t$ ,  $z = t^2$  from  $(0, 1, 0)$  to  $(2\pi, 1, 4\pi^2)$

Q. Find the length of the curve  $x = \sin t - t \cos t$   
 $y = \cos t + t \sin t$ , &  $z = t^2$  from  $(0, 1, 0)$  to  $(-2\pi, 1, 4)$

$$\Rightarrow r(t) = [\sin t - t \cos t] i + [\cos t + t \sin t] j + t^2 k.$$

$$r'(t) = \cos t - (t \times \sin t + \cos t) i + [-\sin t + (t \times \cos t + \sin t)] j + 2t k.$$

$$r'(t) = [\cos t + t \sin t - \cos t] i + [-\sin t + t \cos t + \sin t] j + 2t k$$

$$r'(t) = \underline{\underline{t \sin t i + t \cos t j + 2t k}}$$



$$x(t) = \sin t - t \cos t \quad y(t) = \cos t + t \sin t \quad z(t) = t^2$$

$$x'(t) = t \sin t \quad y'(t) = t \cos t \quad z'(t) = 2t$$

Consider  $(0, 1, 0)$

$$x = \sin t - t \cos t \Rightarrow 0 = \sin t - t \cos t$$

$$\therefore \underline{t=0}$$

$$y = \cos t + t \sin t \Rightarrow 1 = \cos t + t \sin t$$

$$\therefore \underline{t=0}$$

$$z = t^2 \Rightarrow 0 \Rightarrow t^2 \Rightarrow \underline{t=0}$$

$$\therefore \underline{a=0}$$

Consider  $(-2\pi, 1, 4\pi^2)$

$$x = \sin t - t \cos t \Rightarrow -2\pi = \sin t - t \cos t \Rightarrow \underline{t=2\pi}$$

$$y = \cos t + t \sin t \Rightarrow 1 = \cos t + t \sin t \Rightarrow \underline{t=2\pi}$$

$$z = t^2 \Rightarrow 4\pi^2 = t^2 \Rightarrow \sqrt{4\pi^2} = t$$

$$= \underline{2\pi} = t$$

$$\therefore \underline{b=2\pi}$$

$$L = \int_0^{2\pi} \sqrt{(t \sin t)^2 + (t \cos t)^2 + (2t)^2} dt$$

$$= \int_0^{2\pi} \sqrt{t^2 \sin^2 t + t^2 \cos^2 t + 4t^2} dt$$

$$= \int_0^{2\pi} \sqrt{t^2 (\sin^2 t + \cos^2 t) + 4t^2} dt$$

$$= \int_0^{2\pi} \sqrt{t^2 \times 1 + 4t^2} dt$$

$$= \int_0^{2\pi} \sqrt{t^2 + 4t^2} dt$$

$$\begin{aligned}
&= \int_0^{2\pi} \sqrt{5t^2} dt \\
&= \int_0^{2\pi} \sqrt{5} \times \sqrt{t^2} dt \\
&= \sqrt{5} \int_0^{2\pi} \sqrt{t^2} dt \\
&= \sqrt{5} \int_0^{2\pi} (t^2)^{1/2} dt \\
&= \sqrt{5} \int_0^{2\pi} t dt \\
&= \sqrt{5} \left[ \frac{t^2}{2} \right]_0^{2\pi} \\
&= \sqrt{5} \left[ \frac{2\pi^2}{2} - \frac{0^2}{2} \right] \\
&= \sqrt{5} \cdot \left[ \frac{2 \cdot 4\pi^2}{2} - 0 \right] \\
&= \sqrt{5} \left[ 2\pi^2 \right]
\end{aligned}$$

### \* Motion on a curve.

consider a vector valued function  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$  if  $x, y, \& z$  are twice differentiable functions of  $t$ , then the velocity vector ( $\mathbf{v}(t)$ ), acceleration  $\mathbf{a}(t)$  & speed  $\|\mathbf{v}(t)\|$  are defined by.

$$\left. \begin{aligned}
\mathbf{v}(t) &= \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k} \\
\mathbf{a}(t) &= \mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k} \\
\text{Speed} &= \|\mathbf{v}(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}
\end{aligned} \right\}$$

→ The position vector of a particle in space at time  $t$  is,

$$r(t) = (t+1)i + (t^2-1)j + 2tk$$

(a) Find the particle's velocity & acceleration vectors.

(b) Find the particle's speed & direction of motion at  $t=1$

(c) Find the angle between the velocity & acceleration vectors at  $t=1$

⇒ (a)  $r(t) = (t+1)i + (t^2-1)j + 2tk$ .

$$\text{velocity } v(t) = r'(t) = 1i + \underline{2tj} + 2k$$

$$\text{acceleration } a(t) = r''(t) = \underline{2j}$$

$$\begin{aligned} \text{(b) Speed } \|v(t)\| &= \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} \\ &= \sqrt{1^2 + (2t)^2 + (2)^2} \\ &= \sqrt{1 + 4t^2 + 4} \\ &= \sqrt{5 + 4t^2} \end{aligned}$$

at  $t=1$ ,

$$\text{Speed} = \sqrt{5 + 4 \times 1^2}$$

$$= \sqrt{5 + 4} = \sqrt{9} = \underline{\underline{3}}$$

$$\text{Direction} = \frac{v(t)}{\|v(t)\|}$$

$$\begin{aligned} \text{Direction} &= \frac{v(t)}{\|v(t)\|} = \frac{i + 2tj + 2k}{\sqrt{1^2 + (2t)^2 + 2^2}} \\ &= \frac{i + 2tj + 2k}{\sqrt{1 + 4t^2 + 4}} \\ &= \frac{i + 2tj + 2k}{\sqrt{5 + 4t^2}} \end{aligned}$$

$$\begin{aligned} \text{at } t=1, \text{ Direction} &= \frac{i + (2 \times 1)j + 2k}{\sqrt{5 + 4 \times 1^2}} \\ &= \frac{i + 2j + 2k}{\sqrt{9}} = \frac{i + 2j + 2k}{3} \end{aligned}$$

$$(c) \text{ angle } \cos \theta = \frac{v(t) \cdot a(t)}{\|v(t)\| \|a(t)\|}$$

$$\begin{aligned} \cos \theta &= \frac{[i + 2tj + 2k] \cdot [2j]}{\sqrt{5 + 4t^2} \times \sqrt{(2)^2}} \\ &= \frac{(1 \times 0) + (2t \times 2) + (2 \times 0)}{\sqrt{5 + 4t^2} \times \sqrt{4}} = \frac{4t}{2\sqrt{5 + 4t^2}} \end{aligned}$$

at  $t=1$ ,

$$\cos \theta = \frac{4 \times 1}{2\sqrt{5 + 4(1)^2}} = \frac{4}{2\sqrt{5 + 4}} = \frac{4}{2\sqrt{9}} = \frac{4}{2 \times 3} = \frac{4}{6} //$$

→ A person on a hang glider is spiraling upward due to rapidly rising air on a path having position vector

$$r(t) = 3\cos t \mathbf{i} + 3\sin t \mathbf{j} + t^2 \mathbf{k}, \quad 0 \leq t \leq 4\pi.$$

Then evaluate the following:

(a) The velocity & acceleration vectors.

(b) The glider's speed at any time 't'.

(c) The time, when the glider's acceleration is orthogonal to its velocity.

$$\Rightarrow (a) = r(t) = 3\cos t \mathbf{i} + 3\sin t \mathbf{j} + t^2 \mathbf{k}$$

$$\text{velocity} = r'(t) = -3\sin t \mathbf{i} + 3\cos t \mathbf{j} + 2t \mathbf{k}$$

$$(b) \text{ speed} = \|v(t)\| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (2t)^2}$$

$$= \sqrt{9\sin^2 t + 9\cos^2 t + 4t^2}$$

$$= \sqrt{9[\sin^2 t + \cos^2 t] + 4t^2}$$

$$= \sqrt{9 \times 1 + 4t^2}$$

$$= \underline{\underline{\sqrt{9 + 4t^2}}}$$

(c) Given acceleration & velocity is orthogonal.

$$a(t) \cdot v(t) = 0$$

$$\Rightarrow [(-3\sin t \mathbf{i}) + 3\cos t \mathbf{j} + 2t \mathbf{k}] \cdot [-3\cos t \mathbf{i} - 3\sin t \mathbf{j} + 2t \mathbf{k}] = 0$$

$$\Rightarrow q \sin t \cos t - q \sin t \cos t + 4t = 0$$

$$\Rightarrow 4t = 0$$

$$\Rightarrow \underline{t=0}$$

If a particle moves with a constant speed, show that its velocity vector  $v$  is  $\perp$  to its acceleration vector?

$\Rightarrow$  Suppose a particle moves with constant speed. Then speed = constant.

$$\Rightarrow \|v(t)\| = \text{constant} \quad \forall t$$

$$\therefore v(t) \cdot v(t) = \|v(t)\|^2 = C, \text{ a constant.}$$

$$[\because \|v(t)\| = \text{constant}]$$

Differentiating both sides of the above eq<sup>n</sup> with respect to  $t$ , we obtain.

$$\frac{d}{dt} [v(t) \cdot v(t)] = \frac{d}{dt} (C).$$

$$\frac{d}{dt} [v(t) \cdot v(t)] = 0. \quad [\because \frac{d}{dt} \text{constant} = 0]$$

then by product rule,

$$v(t) \cdot \frac{d}{dt} v(t) + v(t) \cdot \frac{d}{dt} v(t) = 0$$

$$2 v(t) \cdot \frac{d}{dt} v(t) = 0$$

$$v(t) \cdot \frac{d}{dt} v(t) = \frac{0}{2} \\ = 0 //$$

$$v(t) \cdot a(t) = 0 \quad \forall t. \quad \left[ \because \frac{d}{dt} v(t) = a(t) \right]$$

Hence  $v$  is  $\perp^{\circ}$  to  $a$ .