1. FERMATS PRINCIPLE

Introduction

Fermats principle is the most important fundamental principle in optics. Fermat's principle also known as the principle of least time, is the link between ray optics and wave optics. First proposed by the French mathematician pierre de Fermat in 1662,

Optical path

The distance S that a light ray travels in any medium is the product of velocity (v) and time (t)

$$S = vt$$

From the definition of refractive index $\mu = c/v$ or $v = c/\mu$

$$S = \frac{c}{\mu} t \qquad \mu S = ct$$

This product μS is called optical path

Optical path is defined as the product of geometrical distance and the refractive index

For example if light travels a distance S_1 in a medium of refractive index μ_1, a distance S_2 in a medium of refractive index μ_2 and so on.

The optical path $\mu_1 S_1 = \mu_2 S_2 = \ldots$

when a ray travels distances S_1, S_2, S_3 etc. in media of refractive indices μ_1, μ_2, μ_3 etc. Then the optical path is given by,

$$S = \mu_1 S_1 + \mu_2 S_2 + \mu_3 S_3 + \cdots = \sum_{i=1} \mu_i S_i$$

For a medium of continuously varying optical density, the optical path of a ray going from P to Q is given by.

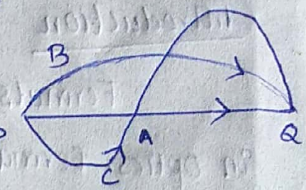
$$S = \int_P^Q \mu\, ds$$

Fermats principle of least time

when a ray travels from one point to another point through a set of media, out of all possible paths it always follows that path along which the time taken is minimum.

Suppose a ray of light travels from P to Q.

There are several possible paths like PAQ, PBQ, PCQ etc.



According to Fermats principle of least time, the ray will choose the path in which the time taken is minimum.

Let dt be the time in which light ray travels a distance ds in the medium.

$$dt = ds/v \qquad v \rightarrow \text{velocity of light in that medium}$$

$$\mu = c/v \Rightarrow v = c/\mu$$

$$dt = \mu/c \, ds$$

time required to travel from P to Q

$$t = \int_P^Q \frac{\mu ds}{c} = \frac{1}{c}\int_P^Q \mu ds$$

$\int_P^Q \mu ds$ is called optical path

Fermats principle of extremum path

When a ray travels from one point to another point through a set of media, out of all possible paths it always follows that path along which the time taken is extremum. (Minimum or Maximum or Stationary) This is called Fermat's principle of extremum path or Fermats principle of stationary time.

Rectilinear propagation of light

According to Fermats principle,

$$\delta \int_P^Q \mu ds = 0$$

Homogeneous and isotropic medium, $\mu \rightarrow$ constant

$$\delta \left[\mu \int_P^Q ds\right] = 0 \quad \text{or} \quad \delta \int_P^Q ds = 0$$

which means that between two points a ray will travel in a straight line and not any other path.

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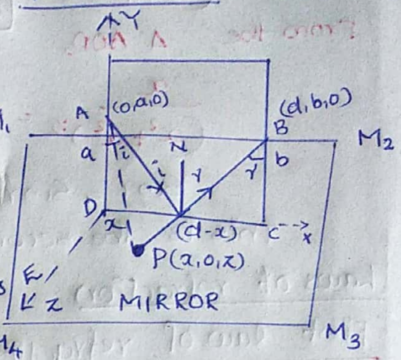
Laws of reflection and refraction from Fermat's principle

First Law of reflection



Consider a plane mirror $M_1M_2M_3M_4$. Let A and B be two points above the M_1 mirror and located in a plane ABCD normal to the plane of the mirror. Light coming from point A is reflected towards B. Suppose the light ray passes through a point P. It means that the path of the ray is along APB.

Draw a plane ABCD normal to the mirror plane $M_1M_2M_3M_4$. Take the origin of coordinates at D. Let DC, DA, DE be the X, Y and z axes. Let DA = a, CB = b and DC = d. The point P has the coordinates $(x, 0, z)$.

We have to minimise only the path length APB = AP+PB instead of optical path length μ APB.

Path length APB, L = AP+PB

$$L = \left[(x-0)^2+(0-a)^2+(z-0)^2\right]^{1/2} + \left[(x-d)^2+(0-b)^2+(z-0)^2\right]^{1/2}$$

$$L = (x^2+a^2+z^2)^{1/2} + \left[(x-d)^2+b^2+z^2\right]^{1/2} \qquad (1)$$

Apply Fermat's principle to get actual path length. The path APB can be varied by varying x and z. We obtain the minimum value of L, that is the shortest path by taking the derivative of L with respect to z and setting the derivative equal to zero.

$$\left(\frac{\partial L}{\partial z}\right)_x = \frac{1}{2}\frac{1}{(x^2+a^2+z^2)^{1/2}}\cdot 2z + \frac{1}{2}\cdot\frac{1}{\left[(x-d)^2+b^2+z^2\right]}\cdot 2z = 0$$

$$z\left\{\frac{1}{(x^2+a^2+z^2)^{1/2}} + \frac{1}{\left[(x-d)^2+b^2+z^2\right]^{1/2}}\right\} = 0$$

As the factor within the brackets cannot be zero, z Must be zero

Second Law of reflection

using $z=0$ in equation (1)

$$L = (x^2+a^2)^{1/2} + \left[(x-d)^2+b^2\right]^{1/2}$$

Now taking the derivative of L with respect to x, and setting the derivative equal to zero.

$$\frac{dL}{dx} = \frac{2x}{2(x^2+a^2)^{1/2}} + \frac{2\cdot(x-d)}{2\left[(x-d)^2+b^2\right]^{1/2}} = 0$$

$$\frac{x}{(x^2+a^2)^{1/2}} = \frac{d-x}{[(x-d)^2+b^2]^{1/2}}$$

From the Δ AOD,

$$\frac{x}{(x^2+a^2)^{1/2}} = \sin i \text{. from the } \Delta BOC \; \frac{d-x}{[(x-d)^2+b^2]^{1/2}} = \sin r$$

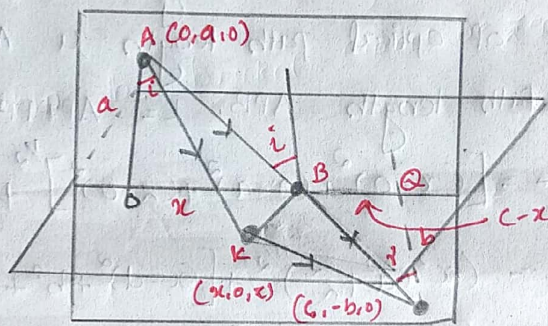
$$\sin i = \sin r \quad \text{or} \quad i = r$$

This is the second law of reflection

Laws of refraction

First law of refraction

Consider a plane surface S separating two media of refractive indices μ_1 and μ_2. Let A and C be two points lying in the two different media. A light ray starts from A and reaches at C. The ray can choose different paths such as ABC or AKC. According to Fermats principle this path will be such that the optical path is minimum. Suppose the ray takes the path AKC.



OA = a

OQ = c

OB = x

BQ = c - x

QC = b

The optical path length through the point K $(x, 0, z)$ is given by

$$L = \mu_1 AK + \mu_2 KC$$

$$L = \mu_1 (x^2+a^2+z^2)^{1/2} + \mu_2 [(x-c)^2+b^2+z^2]^{1/2} \quad \cdots (1)$$

We now apply Fermats principle to get actual path length. The path can be varied by varying x and z. We obtain the minimum value of L that is the shortest optical path by taking the derivative of L with respect to z and setting the derivative equal to zero.

$$\left(\frac{\partial L}{\partial z}\right)_x = \frac{\mu_1 \cdot 2z}{2(x^2+a^2+z^2)^{1/2}} + \frac{\mu_2 \cdot 2z}{2[(x-c)^2+b^2+z^2]^{1/2}} = 0$$

$$z \left\{ \frac{\mu_1}{(x^2+a^2+z^2)^{1/2}} + \frac{\mu_2}{2\left[(x-c)^2+b^2+z^2\right]^{1/2}} \right\} = 0$$

As the factor within the bracket cannot be zero. z must be zero.

Second law of refraction

using $z=0$ in equation (1)

$$L = \mu_1 (x^2+a^2)^{1/2} + \mu_2 \left[(x-c)^2+b^2\right]^{1/2}$$

Now taking the derivative of L with respect to x and Setting the derivative equal to zero.

$$\left(\frac{dL}{dx}\right) = \frac{\mu_1 \cdot 2x}{2(x^2+a^2)^{1/2}} + \frac{\mu_2 \cdot 2(x-c)}{2\left[(x-c)^2+b^2\right]^{1/2}} = 0$$

$$\frac{\mu_1 x}{(x^2+a^2)^{1/2}} = \frac{\mu_2 (c-x)}{\left[(x-c)^2+b^2\right]^{1/2}}$$

From the ΔAOB, $\frac{x}{(x^2+a^2)^{1/2}} = \sin i$

From the ΔBCQ, $\frac{c-x}{\left[(x-c)^2+b^2\right]^{1/2}} = \sin r$

$$\mu_1 \sin i = \mu_2 \sin r$$

$$\frac{\sin i}{\sin r} = \frac{\mu_2}{\mu_1}$$ This is Snell's Law of refraction

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