CHAPTER : 1

VECTOR VALUED FUNCTION

Vector:

A Vector is a quantity that has both magnitude and direction.

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\vec{r}| = \sqrt{x^2 + y^2 + z^2} \Rightarrow$ Modulus of a vector.

Suppose, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ &

$\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

Then $\vec{a} + \vec{b} = (x_1 + x_2)\hat{i} + (y_1 + y_2)\hat{j} + (z_1 + z_2)\hat{k}$

$k\vec{a} = kx_1\hat{i} + ky_1\hat{j} + kz_1\hat{k}$

$\vec{a} - \vec{b} = (x_1 - x_2)\hat{i} + (y_1 - y_2)\hat{j} + (z_1 - z_2)\hat{k}$.

- If a function given in the form of a vector, then it is, called vector-valued function. And it is written by,

$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

• Eg: $r(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$

$r(t) = 2t\,\hat{i} + t\hat{k}$

Limit

Consider a vector valued function $r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

Then L is said to the limit of $r(t)$ at $t = a$ if, $\lim_{t \to a} r(t) = L$.

• Eg: $r(t) = t\hat{i} + e^t\hat{j} - \cos t\,\hat{k}$

Then limit at $t = 0$ is,

$$\lim_{t \to 0} r(t) = \lim_{t \to 0}\left[t\hat{i} + e^t\hat{j} - \cos t\,\hat{k}\right]$$

$$= \lim_{t \to 0} t\hat{i} + \lim_{t \to 0} e^t\,\hat{j} - \lim_{t \to 0} \cos t\,\hat{k}$$

$$= 0\hat{i} + e^0\hat{j} - \cos 0\,\hat{k}$$

$$= 0 + \hat{j} - \hat{k}$$

$$= \underline{\underline{\hat{j} - \hat{k}}}$$

Continuity.

$r(t)$ is said to be continuous at $t = a$

if $\lim_{t \to a} r(t) = r(a)$

• Eg: $r(t) = t\hat{i} - \frac{\sin t}{t}\hat{j} + t^2\hat{k}$

Then continuity at $t=0$ is,

$$\lim_{t\to 0} r(t) = \lim_{t\to 0}\left[t\hat{i} - \frac{\sin t}{t}\hat{j} + t^2\hat{k}\right]$$

$$= \lim_{t\to 0} t\hat{i} - \lim_{t\to 0}\frac{\sin t}{t}\hat{j} + \lim_{t\to 0} t^2\hat{k}$$

$$= 0\hat{i} - \frac{\sin 0}{0}\hat{j} + 0\hat{k}$$

$$= \underline{\underline{-\hat{j}}}$$

over division possible alla

$\therefore \frac{\sin 0}{0}$ = not defined.

& $r(0) = 0i - \frac{\sin 0}{0}\hat{j} + 0\hat{k}$. which is not defined. so $\lim_{t\to 0} r(t) \neq r(0)$

$\therefore$ It is not continuous.

Find the limit of $r(t) = t^2\hat{i} + (3t-2)\hat{j} + (t^3+1)\hat{k}$ at $t=0$.

$\Rightarrow$ So the limit at $t=0$ is,

$$\lim_{t\to 0} r(t) = \lim_{t\to 0}\left[t^2\hat{i} + (3t-2)\hat{j} + (t^3+1)\hat{k}\right]$$

$$= \lim_{t\to 0} t^2\hat{i} + \lim_{t\to 0}(3t-2)\hat{j} + \lim_{t\to 0}(t^3+1)\hat{k}$$

$$= 0\hat{i} + (3\times 0 - 2)\hat{j} + (0+1)\hat{k}$$

$$= 0 + (-2)\hat{j} + \hat{k}$$

$$= -2\hat{j} + \hat{k}$$

Q: find the limit of $r(t) = \sin t\,\hat{i} + \cos t\,\hat{j} + (t+1)\hat{k}$ at $t = \pi/2$

⇒ limit at $t = \pi/2$ is,

$$\lim_{t\to\pi/2} r(t) = \lim_{t\to\pi/2}\left[\sin t\,\hat{i} + \cos t\,\hat{j} + (t+1)\hat{k}\right]$$

$$= \lim_{t\to\pi/2} \sin t\,\hat{i} + \lim_{t\to\pi/2} \cos t\,\hat{j} + \lim_{t\to\pi/2} (t+1)\hat{k}$$

$$= \sin \pi/2\,\hat{i} + \cos \pi/2\,\hat{j} + (\pi/2 + 1)\hat{k}$$

$$= 1\hat{i} + 0\hat{j} + (\pi/2 + 1)\hat{k}$$

$$= \hat{i} + (\pi/2 + 1)\hat{k}$$

Q: check the continuity of $r(t) = t\hat{i} + a\hat{j} + (a^2 - t^2)\hat{k}$ at $t = 0$

⇒ The condinuity at $t = 0$,

$$\lim_{t\to 0} r(t) = \lim_{t\to 0}\left[t\hat{i} + a\hat{j} + (a^2 + t^2)\hat{k}\right]$$

$$= \lim_{t\to 0} t\hat{i} + \lim_{t\to 0} a\hat{j} + \lim_{t\to 0} (a^2 + t^2)\hat{k}$$

$$= 0\hat{i} + a\hat{j} + a^2\hat{k}$$

$$= a\hat{j} + a^2\hat{k}$$

now, $r(0) = 0\hat{i} + a\hat{j} + a^2\hat{k}$

$$= a\hat{j} + a^2 k^2$$

so $\lim_{t \to 0} r(t) = r(a)$

ie; it is condinuous.

Check the condinuity of $r(t) = \cos t\hat{i} + \sin t\hat{j} + t\hat{k}$, at $t = \pi/2$

The condinuity at $t = \pi/2$.

$$\lim_{t \to \pi/2} r(t) = \lim_{t \to \pi/2} [\cos t\hat{i} + \sin t\hat{j} + t\hat{k}]$$

$$= \lim_{t \to \pi/2} \cos t\hat{i} + \lim_{t \to \pi/2} \sin t\hat{j} + \lim_{t \to \pi/2} t\hat{k}$$

$$= \cos \pi/2\, \hat{i} + \sin \pi/2\, \hat{j} + \pi/2\, \hat{k}$$

$$= 0\hat{i} + \hat{j} + \pi/2\, \hat{k}$$

$$= \hat{j} + \pi/2\, \hat{k}$$

and $r(a) = r(\pi/2)$

$$r(\pi/2) = \cos \pi/2\, \hat{i} + \sin \pi/2\, \hat{j} + \pi/2\, \hat{k}$$

$$= 0\hat{i} + \hat{j} + \pi/2\, \hat{k}$$

$$= \hat{j} + \pi/2\, \hat{k}$$

so $\lim_{t \to \pi/2} r(t) = r(a)$

$\therefore$ It is condinuous.

Derivatives

consider a vector-valued function $r(t) = x(t)\hat{i} +$

$y(t)\hat{j} + z(t)\hat{k}$. Then the derivative of the vector valued function is $r'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}$.

Eg: ① $r(t) = \sin t\,\hat{i} + \cos t\,\hat{j} - t^3\hat{k}$

$$r'(t) = \cos t\,\hat{i} - \sin t\,\hat{j} - 3t^2\hat{k}$$

② $r(t) = \sin t\,\hat{i} + t^3\hat{j} - e^t\hat{k}$

$$r'(t) = \cos t\,\hat{i} + 3t^2\hat{j} - e^t\hat{k}$$

concider two vector valued function $r_1(t)$ & $r_2(t)$. Then the derivative of their dot product is,

$$\frac{d}{dt}\left[r_1(t)\cdot r_2(t)\right] = r_1(t)\cdot\frac{d\,r_2(t)}{dt} + r_2(t)\cdot\frac{d\,r_1(t)}{dt}$$

concider two vector valued functions $r_1(t)$ & $r_2(t)$. Then the derivative of their cross product is,

$$\frac{d\left[r_1(t)\times r_2(t)\right]}{dt} = r_1(t)\times\frac{d\,r_2(t)}{dt} + r_2(t)\times\frac{d\,r_1(t)}{dt}$$

Find the derivative of the following;

$r(t) = te^t\hat{i} + \cos t\,\hat{j}$

$$r'(t) = \frac{d\,t\cdot e^t}{dt}\hat{i} + \frac{d\cos t}{dt}\hat{j}$$

applying product rule,

$\therefore \frac{d(t\cdot e^t)}{dt} = t\cdot\frac{de^t}{dt} + e^t \times 1$

$= t\cdot e^t + e^t$

$\therefore r'(t) = t\cdot e^t + e^t\,\hat{i} - \sin t\,\hat{j}$

2] $r(t) = 2t\hat{i} - 3t^2\hat{k}$

$r'(t) = 2\hat{i} - 6t\hat{k}$

3] $r(t) = \cos t\,\hat{i} + e^t\hat{j} + (t\cdot \ln t)\hat{k}$

$= -\sin t\,\hat{i} + e^t\hat{j} + \left[t\cdot\frac{d\ln t}{dt} + \ln t\cdot\frac{dt}{dt}\right]\hat{k}$

$= -\sin t\,\hat{i} + e^t\hat{j} + \left[t\times\frac{1}{t} + \ln t\times 1\right]$

$= \sin t\,\hat{i} + e^t\hat{j} + [1 + \ln t]\hat{k}$

$= \sin t\,\hat{i} + e^t\hat{j} + [\ln t + 1]\hat{k}$

4] $r(t) = 2t\hat{i} + t\hat{j} + 3t\hat{k}$

$r'(t) = 2\hat{i} + \hat{j} + 3\hat{k}$

$\frac{d(\cos 4t)}{dt}$

$= \cancel{\cos\cdot\frac{d\,4t}{dt} + 4t\frac{d}{dt}}\cos$

$= \cancel{\cos}$

5] $r(t) = \cos 4t\,\hat{i} + \sin 4t\,\hat{j} + t\hat{k}$

$= -4\sin 4t\,\hat{i} + 4\cos 4t\,\hat{j} + \hat{k}$

6] $r(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$

$= \hat{i} + 2t\hat{j} + 3t^2\hat{k}$

Let $r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ be vector valued function. Then the integral of vector function is,

$$\int_a^b r(t)\,dt = \int_a^b x(t)\cdot\hat{i} + \int_a^b y(t)\hat{j} + \int_a^b z(t)\hat{k}.$$

$\int x^2 \cdot e^{x} dx =$

Eg:

$r(t) = t^3\hat{i} + 3t^2\hat{j} + t\hat{k}\ ;\ 0 \le t \le 1$

$$\Rightarrow \int_0^1 r(t)\,dt = \int_0^1 [t^3\hat{i} + 3t^2\hat{j} + t\hat{k}]\,dt$$

$$= \int_0^1 t^3 dt\ \hat{i} + \int_0^1 3t^2 dt\ \hat{j} + \int_0^1 t\,dt\ \hat{k}$$

$$= \left[\frac{t^4}{4}\hat{i}\right]_0^1 + \left[\frac{3t^3}{3}\hat{j}\right]_0^1 + \left[\frac{t^2}{2}\hat{k}\right]_0^1$$

$$= \left.\frac{t^4}{4}\hat{i}\right]_0^1 + \left[t^3\hat{j}\right]_0^1 + \left[\frac{t^2}{2}\hat{k}\right]_0^1$$

$$= \left[\frac{1}{4} - 0\right]\hat{i} + (1^3 - 0)\hat{j} + (1/2 - 0)\hat{k}$$

$$= 1/4\,\hat{i} + \hat{j} + 1/2\,\hat{k}$$

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HW

Q: Find the integral of the following?

① $r(t) = t^2 \hat{i} - 2t\hat{j} + t^4 e^t \; ; \; 0 \leq t \leq 2$

$$\Rightarrow \int_0^2 r(t)\,dt = \int_0^2 (t^2\hat{i} - 2t\hat{j} + t^4 e^t)\,dt$$

$$= \int_0^2 t^2\hat{i}\,dt - \int_0^2 2t\hat{j}\,dt + \int_0^2 t^4 e^t\,dt$$

$$= \left[\frac{t^3}{3}\hat{i}\right]_0^2 - \left[\cancel{2}\frac{t^2}{\cancel{2}}\hat{j}\right]_0^2 + \left[\frac{t^5}{5}e^{\cancel{t}}\right]_0^2$$

$$= (8/3 - 0)\hat{i} - (4-0)\hat{j} + (32/5 - 0)$$

$$= 8/3\,\hat{i} - 4\hat{j} + 32/5$$

$\int 1/t$

② $r(t) = \frac{2}{t}\hat{j} + t\hat{k} \; ; \; 1 \leq t \leq 3$

$$\Rightarrow \int_1^3 (r(t))\,dt = \int_1^3 (2/t\,\hat{j} + t\hat{k})\,dt = \left[2\ln t\,j + \frac{t^2}{2}k\right]_1^3$$

$$= \cancel{\int_1^3 2/t\,\hat{i}\,dt + \int_0^3 t\hat{k}\,dt} = \left[2\ln 3\,j + \frac{3^2}{2}k\right] - \left[2\ln 1\,\hat{j} + \frac{1^2}{2}k\right]$$

$$= \cancel{\int_1^3 2 \times \frac{1}{t}\,dt + \int_0^3 t\hat{k}}\,dt$$

$$= \left[2\ln 3\,\hat{j} + \frac{9}{2}k\right] - \left[2\times 0\,\hat{j} + \frac{1}{2}k\right]$$

$$=$$

$$= 2\ln 3\,\hat{j} + \frac{9}{2}k - \left[0\hat{j} + \frac{1}{2}k\right] \quad [\because \ln 1 = 0]$$

$$= 2\ln 3\,\hat{j} + \frac{9}{2}k - \frac{1}{2}k$$

$$= 2\ln 3\,\hat{j} + \left(\frac{9}{2} - \frac{1}{2}\right)k$$

$$= 2\ln 3\,\hat{j} + \frac{8}{2}k = 2\ln 3\,\hat{j} + 4k$$

$r(t) = \cos 4t\,\hat{i} + \sin 4t\,\hat{j} + t\hat{k}, \; 0 \le t \le 2\pi$. (5)

$$\int_0^{2\pi} r(t)\,dt = \int_0^{2\pi} (\cos 4t\,\hat{i} + \sin 4t\,\hat{j} + t\hat{k})\,dt$$

$$= \int_0^{2\pi} (\cos 4t\,\hat{i})\,dt + \int_0^{2\pi} (\sin 4t\,\hat{j})\,dt + \int_0^{2\pi} (t\,\hat{k})\,dt$$

$$= \left.\frac{\sin 4t}{4}\hat{i} + \frac{-\cos 4t}{4}\hat{j} + \frac{t^2}{2}k\right]_0^{2\pi}$$

$$= \left[\frac{\sin 4(2\pi)}{4}i - \frac{\cos 4(2\pi)}{4}j + \frac{2\pi^2}{2}k\right] - \left[\frac{\sin 4\times 0}{4}i - \frac{\cos 4\times 0}{4}j + \frac{0}{2}\right]$$

$$= \left[0i - \frac{1}{4}j + \frac{4\pi^2}{2}k\right] - \left[\frac{0}{4}i - \frac{1}{4}j + 0k\right]$$

$$= -\frac{1}{4}j + \frac{4\pi^2}{2}k + \frac{1}{4}j$$

$$= \frac{4\pi^2}{2}k = \underline{\underline{2\pi^2 k}}$$

(6)

$r(t) = a\cos t\,\hat{j} + a\sin t\,\hat{k}, \; 0 \le t \le 2\pi$

$$\int_0^{2\pi} (r(t))\,dt = \int_0^{2\pi} (a\cos t\,\hat{j} + a\sin t\,\hat{k})\,dt$$

$$= \int_0^{2\pi} (a\cos t\,\hat{j})\,dt + \int_0^{2\pi} (a\sin t\,\hat{k})\,dt$$

$$= a\int_0^{2\pi} (\cos t\,\hat{j})\,dt + a\int_0^{2\pi} (\sin t\,\hat{k})\,dt$$

$$= a\left[\sin x\right]_0^{2\pi}\hat{j} + a\left[-\left[\cos t\right]_0^{2\pi}\right]\hat{k}$$

$$= a\left[\sin 2\pi - \sin 0\right]\hat{j} + a\left[-\cos 2\pi + \cos 0\right]\hat{k}$$

$$= a\left[\sin 2\pi - \sin 0\right]\hat{j} + a\left[-\cos 2\pi + \cos 0\right]\hat{k}$$

$-1+0$

$$= \underline{\underline{0 - 0}} = 0$$

$r(t) = t\hat{i} + \frac{t^2}{2}\hat{j}, \quad 0 \le t \le 2$ $\qquad \frac{x}{2} = \frac{x^2}{2\times 2}$

$$\int_0^2 r(t)\,dt = \int_0^2 (t\hat{i})\,dt + \int_0^2 \left(\frac{t^2}{2}\hat{j}\right)dt$$

$$= \left[\frac{t^2}{2}\right]_0^2 \hat{i} + \left[\frac{t^3}{2\times 3}\right]_0^2 \hat{j}$$

$$= \left[\frac{4}{2} - 0\right]\hat{i} + \left[\frac{8}{6}\right]\hat{j}$$

$$= 2\hat{i} + \frac{4}{3}\hat{j}$$

$\frac{\pi^2}{4} \times \frac{x}{1}$ $\quad \frac{\pi^2}{2} \times (1) - \frac{\pi^2}{2}$ $\quad \frac{\pi^2/4}{2}$ $\quad \pi \; 180$

$r(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}, \quad 0 \le t \le \pi/2$

$$\int_0^{\pi/2} r(t)dt = \int_0^{\pi/2} (\cos t)\hat{i}\,dt + \int_0^{\pi/2} (\sin t)\hat{j}\,dt + \int_0^{2\pi} t\hat{k}\,dt$$

$$= +\sin t\,\hat{i}\Big]_0^{\pi/2} + \cos t\,\hat{j}\Big]_0^{\pi/2} + \frac{t^2}{2}\Big]_0^{\pi/2}\hat{k}$$

$$= \cancel{\left[-\sin \pi/2 \; \sin 0\right]}\hat{i} + \left[\cos \pi/2 - \cos 0\right]\hat{j} + \cancel{\ }^2$$

$$\cancel{\left[\frac{(\pi/2)^2}{2} - 0\right]}\hat{k} \quad \left[\sin \pi/2\, i - \cos \pi/2\, j + \frac{(\pi/2)^2}{2}k\right] - \left[\sin 0\, i - \cos 0\, j + 0^2/2\, k\right]$$

$$= \cancel{\left[\frac{\pi^2}{4} \times \frac{2}{1} - 0\right]}\hat{k} \qquad = \left[1i - 0j + \frac{\pi^2/4}{2}k\right] - \left[0i - 1j + 0k\right]$$

$$= \cancel{\frac{2\pi}{4}} \qquad = i + \pi^2/8\, k + j$$

$$= i + j + \pi^2/8\, k$$

Concider a vector valued function $r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$. and its derivative $r'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}$.

Then the length of a space curve is,

$$\boxed{L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt}$$

Q:

Eg:

Find the length of the curve $r(t) = \langle 6\cos(2t), 6\sin(2t), 5t \rangle$ for $t \in [0, \pi]$?.

$r(t) = 6\cos(2t)\cdot\hat{i} + 6\sin 2t\, j + 5t\, k$

$r'(t) = 6 \times -\sin 2t \times 2i + 6 \times \cos 2t \cdot \times 2j + 5k$

$r'(t) = -12\sin 2t\, i + 12\cos 2t\, \hat{j} + 5k$

here, ~~when~~ $a = 0$, $b = \pi$.

$$\therefore L = \int_0^\pi \sqrt{(-12\sin 2t)^2 + (12\cos 2t)^2 + (5)^2}\, dt$$

$$L = \int_0^\pi \sqrt{144\sin^2 2t + 144\cos^2 2t + 25}\, dt$$

$$= \int_0^\pi \sqrt{144[\sin^2 2t + \cos^2 2t] + 25}\, dt$$

$$= \int_0^\pi \sqrt{144 \times 1 + 25}\, dt$$

$$= \int_0^{\pi} \sqrt{169}\, dt$$

$$= \int_0^{\pi} 13\, dt$$

$$= 13t\Big]_0^{\pi} = \underline{\underline{13\pi}}$$

HW (Find the length of the following)

① $r(t) = \langle 4\cos t\hat{i},\ 4\sin t\hat{j},\ 3t\hat{k} \rangle\ [0,1]$

$$\therefore r(t) = 4\cos t\hat{i} + 4\sin t\hat{j} + 3t\hat{k}$$

$$r'(t) = 4\sin t i + 4\cos j + 3k$$

$$= \underline{\underline{-4\sin t i + 4\cos j + 3k}}$$

$a = 0$ & $b = 1$

$$\therefore L = \int_0^1 \sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2}\, dt$$

$$= \int_0^1 \sqrt{16\sin^2 t + 16\cos^2 t + 9}\, dt$$

$$= \int_0^1 \sqrt{16(\sin^2 t + \cos^2 t) + 9}\, dt$$

$$= \int_0^1 \sqrt{16 \times 1 + 9}\, dt$$

$$= \int_0^1 \sqrt{25}\, dt$$

$$= \int_0^1 5\, dt$$

$$= 5t\Big]_0^1 = 5 \times 1 - 0 = \underline{\underline{5}}.$$

② $r(t) = t\hat{i} + \cosh t j$

$r'(t) = i + \sinh t j$

$x'(t) = 1, \quad y'(t) = \sinh t$

$a = 0, \quad b = 1$

$L = \int_0^1 \sqrt{1^2 + (\sinh t)^2}\, dt$

$= \int_0^1 \sqrt{1 + \sinh^2 t}\, dt$

$= \int_0^1 \sqrt{\cosh^2 t}\, dt = \int_0^1 \cosh t\, dt$

$= \sinh t \Big]_0^1 = \sinh 1 - \sinh 0$

$= \sinh 1 \qquad [\because \sinh 0 = 0]$

$r(t) = \cos t i + \sin j + tk; \quad t \in [0, 2\pi]$

$r'(t) = \cancel{\cos t i} \cdot -\sin t i + \cos j + 1k$

$a = 0 \quad b = 2\pi$

$\therefore L = \int_0^{2\pi} \sqrt{-\sin t i + \cos j + 1k}\, dt$

$= \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1}\, dt$

$= \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1}\, dt$

$= \int_0^{2\pi} \sqrt{1+1}\, dt$

$= \int_0^{2\pi} \sqrt{2}\, t = \sqrt{2} \cdot 2\pi$

5

3) $\gamma(t) = ti + \sin 2\pi t\, j + \cos 2\pi t\, \hat{k}, \quad t \in [0,1]$

$\gamma'(t) = \cancel{i + \cos 2\pi t\, j - 2\sin 2\pi t}$

$= i + \cos 2\pi t\, j \times 2\pi j + -\sin 2\pi t \times 2\pi k$

$\gamma'(t) = \underline{\underline{i + 2\pi \cos 2\pi t\, j - 2\pi \sin 2\pi t\, k}}$

$x'(t) = 1, \quad y'(t) = 2\pi \cdot \cos 2\pi t, \quad z'(t) = -2\pi \sin 2\pi t$

$a = 0, \; b = 1$

$\therefore L = \int_0^1 \sqrt{(1)^2 + (2\pi \cos 2\pi t)^2 + (2\pi \sin 2\pi t)^2}\, dt$

$= \int_0^1 \sqrt{1 + 4\pi^2 \cos^2 2\pi t + 4\pi^2 \sin^2 2\pi t}\, dt$

$= \int_0^1 \sqrt{1 + 4\pi^2 (\cos^2 2\pi t + \sin^2 2\pi t)}\, dt$

$= \int_0^1 \sqrt{1 + 4\pi^2 \times 1}\, dt$

$= \int_0^1 \sqrt{1 + 4\pi^2}\, dt$

$= \sqrt{1 + 4\pi^2} \int_0^1 dt$

$= \sqrt{1 + 4\pi^2} \times [t]_0^1$

$= \sqrt{1 + 4\pi^2} \times 1$

$= \underline{\underline{\sqrt{1 + 4\pi^2}}}$

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5) $\gamma(t) = 2\cos t\, i + 2\sin t\, j + \sqrt{5} t\, k, \quad 0 \leq t \leq \pi$

$\gamma'(t) = -2\sin t\, i + 2\cos t\, j + \sqrt{5}\, k$

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$x'(t) = -2\sin t \quad , \quad y'(t) = 2\cos t \quad , \quad z'(t) = \sqrt{5}$

$a = 0 , \; b = \pi$

$$\therefore L = \int_0^{\pi} \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (\sqrt{5})^2} \; dt$$

$$= \int_0^{\pi} \sqrt{(-2)^2 \times \sin^2 t + 2^2 \times \cos^2 t + 5} \; dt$$

$$= \int_0^{\pi} \sqrt{4 \times \sin^2 t + 4\cos^2 t + 5} \; dt$$

$$= \int_0^{\pi} \sqrt{4\left[\sin^2 t + \cos^2 t\right] + 5} \; dt$$

$$= \int_0^{\pi} \sqrt{4 \times 1 + 5} \; dt$$

$$= \int_0^{\pi} \sqrt{9} \; dt \qquad 7)$$

$$= \int_0^{\pi} 3 \, dt$$

$$= 3\pi - 0 = \underline{\underline{3\pi}}$$

$r(t) = ti + (2/3)\, t^{3/2} k \; ; \; 0 \le t \le 8.$

$r'(t) = i + 2/3 \times 3/2 \; t^{1/2}$

$= \underline{\underline{i + t^{1/2} k}}$

$x'(t) = 1 \; , \; y'(t) = t^{1/2}.$

$a = 0 , \; b = 8 \qquad 8)$

$$L = \int_0^{8} \sqrt{(1)^2 + (t^{1/2})^2} \; dt$$

$$= \int_0^{8} \sqrt{1 + t} \; dt.$$

$= \int_0^8 (1+t)^{1/2}\, dt$

$= \left[\dfrac{(1+t)^{3/2}}{3/2} \right]_0^8 \quad \left[\because \int x^n = \dfrac{x^{n+1}}{n+1} \right]$

$= \frac{2}{3}(1+t)^{3/2} \Big]_0^8$

$= \frac{2}{3}\left[(1+8)^{3/2} - (1+0)^{3/2} \right]$

$= \frac{2}{3}\left[9^{3/2} - 1^{3/2} \right]$

$= \frac{2}{3}\left[27 - 1 \right] = \frac{2}{3} \times 26 = \frac{52}{3}$

---

7) $r(t) = (\underset{\text{constant}}{2}+t)i - (t+1)j + tk \;;\; 0 \le t \le 3.$

$r'(t) = \dot{i} - \dot{j} + k$

$x'(t) = 1, \; y'(t) = -1, \; z'(t) = 1$

$a = 0, \; b = 3$

$\therefore L = \int_0^3 \sqrt{1^2 + (-1)^2 + 1^2}\, dt$

$= \int_0^3 \sqrt{1+1+1}\, dt$

$= \int_0^3 \sqrt{3}\, dt$

$= \sqrt{3}t \Big]_0^3 \Rightarrow \left[\sqrt{3} \times 3 - \sqrt{3} \times 0 \right] = 3\sqrt{3}$

---

8) $r(t) = t\cos t\, i + t\sin t\, j + \left[\frac{2\sqrt{2}}{3}\right] t^{3/2} k$

$r'(t) = [t \times -\sin t + \cos t]i + [t\cos t + \sin t]j + \left[\frac{2\sqrt{2}}{3} \times \frac{3}{2} t^{1/2} k\right]$

$= [-t\sin t + \cos t]i + [t\cos t + \sin t]j + \sqrt{2}\, t^{1/2} k$

$x'(t) = -t\sin t + \cos t$, $y'(t) = t\cos t + \sin t$, $z'(t) = \sqrt{2}t^{1/2}$

$a = 0$, $b = \pi$.

$$L = \int_0^\pi \sqrt{(-t\sin t + \cos t)^2 + (t\cos t + \sin t)^2 + (\sqrt{2}t^{1/2})^2}\, dt$$

$$= \int_0^\pi \sqrt{[t^2\sin^2 t + \cos^2 t - 2t\sin t\cos t] + t^2\cos^2 t + \sin^2 t + 2t\cos t\sin t] + 2t}\, dt$$

$$= \int_0^\pi \sqrt{t^2 \sin^2 t + \cos^2 t + t^2\cos^2 t + \sin^2 t + 2t}\, dt$$

$$= \int_0^\pi \sqrt{t^2\sin^2 t + t^2\cos^2 t + \cos^2 t + \sin^2 t + 2t}\, dt$$

$$= \int_0^\pi \sqrt{t^2[\sin^2 t + \cos^2 t] + [\cos^2 t + \sin^2 t] + 2t}\, dt$$

$$= \int_0^\pi \sqrt{t^2 \times 1 + 1 + 2t}\, dt = \int_0^\pi (t^2 + 2t + 1)^{1/2}\, dt$$

$$= \left. \frac{\frac{(t^2+2t+1)^{3/2}}{3/2}}{2t+2} \right]_0^\pi = \left. \frac{(t^2+2t+1)^{3/2}}{3/2} \times \frac{1}{2t+2} \right]_0^\pi$$

$$= \left. \frac{2}{3}[t^2+2t+1]^{3/2} \times \frac{1}{2t+2} \right]_0^\pi$$

$$= \left[\frac{2}{3} \frac{[\pi^2 + 2\pi + 1]}{2\pi + 2}\right] - \left[\frac{2}{3} \frac{[0^2 + 2\times 0 + 1]}{2\times 0 + 2}\right]$$

$$= \frac{2}{3} \frac{(\pi^2 + 2\pi + 1)}{2\pi + 2} - \frac{\frac{2}{3}[1]}{2}$$

$$= \frac{2}{3} \times \frac{\pi^2 + 2\pi + 1}{2\pi + 2} - \frac{2/3}{2}$$

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$$= \frac{2}{3}\left[\frac{2\pi^2+2\pi}{4\pi+4}\right]$$

$$= \frac{2}{3}\left[\frac{2(\pi^2+\pi)}{4(\pi+1)}\right]$$

$$= \frac{1}{3}\left[\frac{\pi^2+\pi}{\pi+1}\right]$$

9) $r(t) = \cos^3 t\, j + \sin^3 t\, k \;;\; 0 \le t \le \pi/2.$

$r'(t) = 3\cos^2 t \times -\sin t\, j + 3\sin^2 t \times \cos t\, k.$

$r'(t) = -3\cos^2 t \cdot \sin t\, j + 3\sin^2 t \cdot \cos t\, k$

$\therefore x'(t) = -3\cos^2 t \sin t \;,\; y'(t) = 3\sin^2 t \cdot \cos t.$

$a = 0 \;,\; b = \pi/2$

$$\therefore L = \int_0^{\pi/2} \sqrt{(-3\cos^2 t \cdot \sin t)^2 + (3\sin^2 t \cdot \cos t)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cdot \cos^2 t}\, dt.$$

$$= \int_0^{\pi/2} \sqrt{9[\cos^4 t \cdot \sin^2 t + \sin^4 t \cdot \cos^2 t]}\, dt$$

$$= \int_0^{\pi/2} \sqrt{9\cos^4 t \cdot \sin^2 t\,[\cos^2 t + \sin^2 t]}\, dt$$

$$= \int_0^{\pi/2} \sqrt{9\cos^2 t \sin^2 t \times 1}\, dt$$

$$= \int_0^{\pi/2} \sqrt{9\cos^2 t \sin^2 t}\, dt$$

$$= \int_0^{\pi/2} 3\cos t \sin t\, dt.$$

$= 3\left[\cancel{\cos t \; \sin t}\right]_0^{\pi/2}$

$= 3\left[\cancel{(\cos \pi/2 \; \sin \pi/2) - (\cos 0 \cdot \sin 0)}\right]$

$= 3\left[\cancel{(0\times 1) - (1\times 0)}\right]$

$= \cancel{3\times 0 = 0}$

$= 3\int_0^{\pi/2} \cos t \sin t \, dt$

$= 3\int_0^1 u\,du = 3\times \frac{u^2}{2}\Big]_0^1$

$\sin t = u$

$= 3\times\left(\frac{1-0}{2}\right) = \frac{3}{2}$ //

$\cos t \, dt = du$

$u = \sin 0 = 0$

$u = \sin \pi/2 = 1$

$r(t) = e^t \cos t\, i + e^t \sin t\, j \quad 0 \le t \le 1$

$r'(t) = \cos t\, e^t i + \sin t \cdot e^t j$

$r'(t) = [\cos t \cdot e^t + e^t \cdot -\sin t]\, i + [\sin t \cdot e^t + e^t \cdot \cos t]\, \hat{j}$ ☆

$r'(t) = [e^t \cos t - \sin t\, e^t]\,\hat{i} + [e^t \sin t + e^t \cos t]\, j$

$x'(t) = e^t \cos t - \sin t\, e^t, \; y'(t) = e^t \sin t + e^t \cos t$

$a = 0, \; b = 1$

$\therefore L = \int_0^1 \sqrt{[e^t \cos t - e^t \sin t]^2 + [e^t \sin t + e^t \cos t]^2}\, dt$

$= \int_0^1 \sqrt{[e^{2t}\cos^2 t + e^{2t}\sin^2 t - 2e^{2t}\cos t \sin t] + [e^{2t}\sin^2 t}$

$\overline{+ e^{2t}\cos^2 t + 2e^{2t}\sin t \cos t}\; dt$

Q:

$$= \int_0^1 \sqrt{e^{2t}[\cos^2 t + \sin^2 t] + e^{2t}[\sin^2 t + \cos^2 t]}\, dt$$

$$= \int_0^1 \sqrt{e^{2t} \times 1 + e^{2t} \times 1}\, dt$$

$$= \int_0^1 \sqrt{e^{2t} + e^{2t}}\, dt$$

$$= \int_0^1 \sqrt{2e^{2t}}\, dt$$

$$= \int_0^1 \sqrt{2} \times \sqrt{e^{2t}}\, dt$$

$$= \int_0^1 \sqrt{2} \times (e^{2t})^{1/2}\, dt = \sqrt{2} \int_0^1 e^{2t \times 1/2}\, dt$$

$$= \sqrt{2} \int_0^1 e^t\, dt =$$

$$= \sqrt{2}\, e^t \Big]_0^1$$

$$= \sqrt{2}\,[e^1 - e^0] = \underline{\underline{\sqrt{2}\,[e-1]}}.$$

☆ **Arc length as a Parameter.**

If a curve $\{r(t)\}$ is already given in terms of the parameter t, the arc length parameter with base point $p(t_0)$ is given by,

$$\boxed{s(t) = \int_{t_0}^{t} \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\, du}$$

eg:-

Q: Find the arc length parameterization of the helix, $r(t) = \cos t\, i + \sin t\, j + tk$, $0 \le t$

$\Rightarrow r(t) = \cos t\, i + \sin t\, j + tk$

$r'(t) = -\sin t\, i + \cos t\, j + k$

$r'(u) = -\sin u\, i + \cos u\, j + k.$

Here $t_0 = 0.$

$\therefore S(t) = \int_0^t \sqrt{(-\sin u)^2 + (\cos u)^2 + (1)^2}\, du$

$S = \int_0^t \sqrt{\sin^2 u + \cos^2 u + 1}\; du$

$S = \int_0^t \sqrt{1+1}\; du$

$= \int_0^t \sqrt{2}\, du$

$= \sqrt{2} \int_0^t du$

$= \sqrt{2}\,[u]_0^t \Rightarrow \sqrt{2}\,[t-0]$

$S = \sqrt{2}\, t$

So, $S = \sqrt{2}\, t$

$\therefore t = \frac{s}{\sqrt{2}}$

$\therefore r(t) = r\left(\frac{s}{\sqrt{2}}\right) =$

$= \cos\left(\frac{s}{\sqrt{2}}\right) i + \sin\left(\frac{s}{\sqrt{2}}\right) j + \left(\frac{s}{\sqrt{2}}\right) k$

This is the arc length ~~a~~ parameterization of the given curve.

Q: $r(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j}, \quad 0 \le t \le 2\pi.$

$r'(t) = -2\sin t\, i + 2\cos t\, j$

$r'(u) = -2\sin u\, i + 2\, \cancel{\sin}\, \cos u\, j$

Here, $t_0 = 0$

$$\therefore S(t) = \int_0^t \sqrt{(-2\sin u)^2 + (2\cos u)^2}\, du$$

$$S = \int_0^t \sqrt{4\sin^2 u + 4\cos^2 u}\, du$$

$$= \int_0^t \sqrt{4(\sin^2 u + \cos^2 u)}\, du$$

$$= \int_0^t \sqrt{4 \times 1}\, du$$

$$= \sqrt{4} \int_0^t du$$

$$= 2 \int_0^t du$$

$$= 2[u]_0^t \Rightarrow 2[t - 0]$$

$$S = 2t$$

So $S = 2t$

$t = \frac{S}{2}$

$$\therefore r(t) = r\left(\frac{S}{2}\right)$$

$$= 2\cos\left(\frac{S}{2}\right) i + 2\sin\left(\frac{S}{2}\right) \hat{j}$$

Q: $r(t) = 4\cos t\, i + 4\sin t\, j + 3t\, k, \quad 0 \le t \le \pi/2$

$r'(t) = -4\sin t\, i + 4\cos t\, j + 3k$

$r'(u) = -4\sin u\, i + 4\cos u\, j + 3k.$

Here, $t_0 = 0$

$$\therefore S(t) = \int_0^t \sqrt{(-4\sin u)^2 + (4\cos u)^2 + 3^2 (5)^2}\, du$$

$$S = \int_0^t \sqrt{16\sin^2 u + 16\cos^2 u + 9}\, du$$

$$= \int_0^t \sqrt{16(\sin^2 u + \cos^2 u) + 9}\, du$$

$$= \int_0^t \sqrt{16 \times 1 + 9}\, du.$$

$$= \int_0^t \sqrt{25}\, du$$

$$= 5\int_0^t du \Rightarrow 5\,[u]_0^t$$

$$= 5\,[t - 0]$$

$$S = \underline{\underline{5t}}$$

So, $S = 5t.$

$$\underline{\underline{t = \frac{S}{5}}}$$

$$\therefore r(t) = r\left(\frac{S}{5}\right) = 4\cos\left(\frac{S}{5}\right) i + 4\sin\left(\frac{S}{5}\right) j + 3\left(\frac{S}{5}\right)$$

$r(t) = e^t \cos t\, i + e^t \sin t\, j + e^t$, Q: $\gamma(t) = e^t \cos t\ \hat{i} + e^t \sin t\ j$

$\gamma'(t) = (\cos t \times e^t + e^t \times -\sin t)i + (\sin t \times e^t + e^t \times \cos t)j$

$\gamma'(t) = (e^t \cos t - e^t \sin t)i + [e^t \sin t + e^t \cos t]j$

$\gamma'(u) = (e^u \cos u - e^u \sin u)i + (e^u \sin u + e^u \cos u)j$

$t_0 = 0$

$s(t) = \int_0^t \sqrt{(e^u \cos u - e^u \sin u)^2 + (e^u \sin u + e^u \cos u)^2}\, du$

$= \int_0^t \sqrt{[(e^u \cos u)^2 + (e^u \sin u)^2 - 2e^u \cos u \times e^u \sin u] + [(e^u \sin u)^2 + (e^u \cos u)^2 + 2e^u \sin u \times e^u \cos u]}\, du$

$= \int_0^t \sqrt{e^{u^2} \cos^2 u + e^{u^2} \sin^2 u + e^{u^2} \sin^2 u + e^{u^2} \cos^2 u}\, du$

$= \int_0^t \sqrt{e^{2u}[\cos^2 u + \sin^2 u] + e^{2u}[\sin^2 u + \cos^2 u]}\, du$

$= \int_0^t \sqrt{e^{2u} \times 1 + e^{2u} \times 1}\, du$

$= \int_0^t \sqrt{2e^{2u}}\, du$

$= \int_0^t \sqrt{2} \cdot \sqrt{e^{2u}}\, du$

$= \sqrt{2} \int_0^t \sqrt{e^{2u}}\, du$

$= \sqrt{2} \times \int_0^t (e^{2u})^{1/2} du$

$= \sqrt{2} \int_0^t e^{\cancel{2}u \times 1/\cancel{2}}\, du$

$= \sqrt{2} \int_0^t e^u\, du$

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$$= \sqrt{2} \times e^{u} \Big]_0^t$$

$$= \sqrt{2} \left[e^{t} - e^{0} \right]$$

$$S = \sqrt{2} \left[e^{t} - 1 \right]$$

$\therefore$

Q: $r(t) = e^t \cdot \cos t \; i + e^t \sin t \; j + e^t k \quad , -\ln 4 \leq t \leq 0$

$$r'(t) = (\cos t \times e^t + e^t \times -\sin t)\, i + (\sin t \times e^t + e^t \cdot \cos t)\, j$$
$$+ e^t k$$

$$r'(t) = (e^t \cos t - e^t \sin t)\, i + (e^t \sin t + e^t \cos t)\, j + e^t k$$

$$r'(u) = (e^u \cos u - e^u \sin u)\, i + (e^u \sin u + e^u \cos u)\, j + e^u k$$

$t_0 = -\ln 4.$

$$\therefore S(t) = \int_{-\ln 4}^{t} \sqrt{(e^u \cos u - e^u \sin u)^2 + (e^u \sin u + e^u \cos u)^2 + (e^u)^2}$$

$$= \int_{-\ln 4}^{t} \sqrt{\left[(e^u \cos u)^2 + (e^u \sin u)^2 - 2\, e^u \cos u \times e^u \sin u\right] + \left[(e^u \sin u)^2 + (e^u \cos u)^2 + 2 e^u \sin u \times e^u \cos u\right] + (e^u)^2}$$

$$= \int_{-\ln 4}^{t} \sqrt{\left[e^{2u} \cos^2 u + e^{2u} \sin^2 u\right] + \left[e^{2u} \sin^2 u + e^{2u} \cos^2 u\right] + e^{2u}}\; du$$

$$= \int_{-\ln 4}^{t} \sqrt{e^{2u} (\cos^2 u + \sin^2 u) + e^{2u} (\sin^2 u + \cos^2 u) + e^{2u}}$$

$$= \int_{-\ln 4}^{t} \sqrt{e^{2u}\times 1 + e^{2u}\times 1 + e^{2u}}\, du.$$

$$= \int_{-\ln 4}^{t} \sqrt{e^{2u} + e^{2u} + e^{2u}}\, du$$

$$= \int_{-\ln 4}^{t} \sqrt{3e^{2u}}\, du$$

$$= \sqrt{3} \cdot \int_{-\ln 4}^{t} \sqrt{e^{2u}}\, du.$$

$$= \sqrt{3} \int_{-\ln 4}^{t} (e^{2u})^{1/2}\, du.$$

$$= \sqrt{3} \int_{-\ln 4}^{t} e^{\cancel{2}u \times 1/\cancel{2}}\, du$$

$$= \sqrt{3} \int_{-\ln 4}^{t} e^{u}\, du.$$

$$= \sqrt{3} \cdot \left[e^{u}\right]_{-\ln 4}^{t}$$

$$= \cancel{\sqrt{3}} \cdot \cancel{\left[e^{-\ln 4}\right]}$$

$$= \sqrt{3} \cdot \left[e^{t} - e^{-\ln 4}\right]$$

Q. Find the length of the circular helix.
$r(t) = a\cos t\, i + a\sin t\, j + ct\, k$
from $(a, 0, 0)$ to $(a, 0, 2\pi c)$?

$$\Rightarrow\ L = \int_{a}^{b} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt \longrightarrow (1)$$

$r(t) = a\cos t\, i + a\sin t\, j + ct\, k$

$r'(t) = -a\sin t\, i + a\cos t\, j + ck.$

$x = a\cos t, \quad y = a\sin t \quad z = ct.$

$x'(t) = -a\sin t \quad y'(t) = a\cos t \quad z'(t) = c$

Now concider the point $(a, 0, 0)$.

$x = a\cos t \Rightarrow a = a\cos t \Rightarrow \cos t = 1 \Rightarrow t = \cos^{-1}(1) = 0$

$y = a\sin t \Rightarrow 0 = a\sin t \Rightarrow \sin t = 0 \Rightarrow t = \sin^{-1}(0) = \underline{\underline{0}}$

$z = ct \Rightarrow 0 = ct \Rightarrow t = \underline{\underline{0}}$

$\therefore \underline{\underline{a = 0}}$

concider $(\overset{x}{a}, \overset{y}{0}, \overset{z}{2\pi c})$

$x = a\cos t \Rightarrow a = a\cos t \Rightarrow 1 = \cos t \Rightarrow t = \cos^{-1}(1) = 0$

$y = a\sin t \Rightarrow 0 = a\sin t \Rightarrow 0 = \sin t \Rightarrow t = \sin^{-1}(0) = \underline{\underline{0}}$

$z = ct \Rightarrow 2\pi c = ct \Rightarrow \underline{\underline{2\pi = t}}$.

$\therefore \underline{\underline{b = 2\pi}}$

$$\therefore L = \int_0^{2\pi} \sqrt{(-a\sin t)^2 + (a\cos t)^2 + c^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t + c^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{a^2(\sin^2 t + \cos^2 t) + c^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{a^2 + c^2}\, dt$$

$$= \sqrt{a^2 + c^2} \int_0^{2\pi} dt$$

$$= \sqrt{a^2 + c^2} \times t \Big]_0^{2\pi}$$

$= (2\pi - 0)\sqrt{a^2+c^2}$

$L = 2\pi\sqrt{a^2+c^2}$

Q: Find the length of the semi cubical parabola.
$r(t) = ti + t^{3/2} j$ from $(0,0,0)$ to $(4,8,0)$?

$\Rightarrow$ $L = \int_a^b \sqrt{(x'(t))^2 + [y'(t)]^2 + [z'(t)]^2} \cdot dt.$

$\therefore r(t) = ti + t^{3/2} j.$

$r'(t) = i + 3/2 \cdot t^{1/2} j.$

$x = t$ $\quad y = t^{3/2}$

$x' = 1$ $\quad y' = 3/2 t^{1/2}$

Now concider the point, $(0,0,0)$

$x = t \Rightarrow 0 = t \Rightarrow t = 0$

$y = t^{3/2} \Rightarrow 0 = t^{3/2} \Rightarrow t = 0.$

$\therefore a = 0$

Concider $(4,8,0)$

$x = t \Rightarrow 4 = t \Rightarrow t = 4$

$y = t^{3/2} \Rightarrow 8 = t^{3/2} \Rightarrow t^{3/2} = 8 \Rightarrow (t^{3/2})^{2/3} = 8^{2/3}$

$= t = 4$

$\therefore b = 4$

$t^{3/2} = 8$

$(t^{3/2})^{2/3} = 8^{2/3}$

$t = 8^{2/3} \Rightarrow \sqrt[3]{8^2}$

$t = 4$ $\quad = \sqrt[3]{64}$

$$\therefore L = \int_0^4 \sqrt{1^2 + \left[\tfrac{3}{2} t^{1/2}\right]^2}\, dt$$

$$= \int_0^4 \sqrt{1 + \tfrac{9}{4}(t^{1/2})^2}\, dt = \int_0^4 \sqrt{1 + \tfrac{9}{4} t}\, dt.$$

$$= \int_0^4 \left(1 + \tfrac{9t}{4}\right)^{1/2} dt.$$

$$= \left. \frac{\dfrac{\left[1 + \frac{9t}{4}\right]^{3/2}}{3/2}}{9/4} \right]_0^t = \left. \frac{\frac{2}{3} \times \left[1 + \frac{9t}{4}\right]^{3/2}}{\frac{9}{4}} \right]_0^4$$

$$= \frac{2}{3} \times \frac{4}{9} \times \left[\left[1 \times \frac{9t}{4}\right]^{3/2}\right]_0^4$$

$$= \frac{8}{27}\left[\left(1 + \frac{9 \times 4}{4}\right)^{3/2} - \left(1 + \frac{9 \times 0}{4}\right)^{3/2}\right]$$

$$= \frac{8}{27}\left[\left(1 + \frac{36}{4}\right)^{3/2} - (1 + 0)^{3/2}\right]$$

$$= \frac{8}{27}\left[(1+9)^{3/2} - (1)^{3/2}\right]$$

$$= \frac{8}{27}\left[(10)^{3/2} - 1\right] //$$

Q: Find the length of the curve $x = t, y = 2t+5,$ $z = 3t$ from $(0, 5, 0)$ to $(1, 7, 3)$

$\Rightarrow$ $r(t) = t\hat{i} + 2t+5\,j + 3t\,k.$

$r'(t) = i + 2j + 3k.$

$x = t \qquad y = 2t+5 \qquad z = 3t$

$x'(t) = 1 \qquad y'(t) = 2 \qquad z'(t) = 3.$

Now concider the point, $(0, 5, 0)$

$x = t \Rightarrow 0 = t \Rightarrow \underline{\underline{t=0}}$

$y = 2 \Rightarrow 5 = 2 \Rightarrow t = 0$ //

$z = 3 \Rightarrow 0 = 3 = \underline{\underline{t=0}}$

$\therefore \underline{\underline{a = 0}}$

Concider $(1, 7, 3)$

$x = t \Rightarrow 1 = t \Rightarrow t = 1$

$y = 2t+5 \Rightarrow 7 = 2t+5 \Rightarrow 2t+5 = 7 = 2t = 7-5$

$\cancel{y = 2 \Rightarrow 7 = 2 \Rightarrow t = 0}$ $\quad t = \frac{2}{2}$

$z = 3t \Rightarrow 3 = 3t \Rightarrow \underline{\underline{t = 3/3 = 1}}$ $\quad \underline{\underline{t = 1}}$.

$\therefore \underline{\underline{b = 1}}$

$$\therefore L = \int_0^1 \sqrt{(1)^2 + (2)^2 + (3)^2}\, dt$$

$$= \int_0^1 \sqrt{1+4+9}\, dt$$

$$= \int_0^1 \sqrt{14}\, dt$$

$$= \sqrt{14} \int_0^1 dt$$

$$= \sqrt{14}\,[1-0]$$

$$= \sqrt{14} \times 1 = \underline{\underline{\sqrt{14}}}.$$

~~Q: Find the length of the curve $x = \sin t - t\cos t$, $y = \cos t + t\sin t$, $z = t^2$ from $(0,1,0)$ to $(2\pi, 1, 4\pi^2)$~~

Q. Find the length of the curve $x = \sin t - t\cos$
$y = \cos t + t\sin t$, & $z = t^2$ from $(0,1,0)$ to $(-2\pi, 1, 4$

$\Rightarrow$ $r(t) = [\sin t - t\cos t]\, i + [\cos t + t\sin t]\, j + t^2 k.$

$r'(t) = \cos t - (t \times -\sin t + \cos t)]\, i + [-\sin t + (t \times \cos t + \sin$
$j + 2tk.$

$r'(t) = [\cos t + t\sin t - \cos t]\, i + [-\sin t + t\cos t + \sin t]\, j + 2t$

$r'(t) = t\sin t\, i + t\cos t\, j + 2tk$

$x(t) = \sin t - t\cos t \quad y(t) = \cos t + \sin t \quad z(t) = t^2$

$x'(t) = t\sin t \qquad y'(t) = t\cos t \qquad z'(t) = 2t.$

Concider $(0, 1, 0)$

$x = \sin t - t\cos t \Rightarrow 0 = \sin t - t\cos t$

$\therefore \underline{\underline{t=0}}$

$y = \cos t + t\sin t \Rightarrow 1 = \cos t + t\sin t.$

$\therefore \underline{\underline{t=0}}$

$z = t^2 \Rightarrow 0 \Rightarrow t^2 \Rightarrow \underline{\underline{t=0}}$

$\therefore \underline{\underline{a=0}}$

Concider $(-2\pi, 1, 4\pi^2)$

$x = \sin t - t\cos t. \Rightarrow -2\pi = \sin t - t\cos t \Rightarrow t = \underline{\underline{2\pi}}$

$y = \cos t + t\sin t \Rightarrow 1 = \cos t + t\sin t \Rightarrow t = \underline{\underline{2\pi}}$

$z = t^2 \Rightarrow 4\pi^2 = t^2 \Rightarrow \sqrt{4\pi^2} = t$

$= \underline{\underline{2\pi}} = t$

$\therefore \underline{\underline{b = 2\pi}}$

$$L = \int_0^{2\pi} \sqrt{(t\sin t)^2 + (t\cos t)^2 + (2t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{t^2\sin^2 t + t^2\cos^2 t + 4t^2}\, dt.$$

$$= \int_0^{2\pi} \sqrt{t^2(\sin^2 t + \cos^2 t) + 4t^2}\, dt.$$

$$= \int_0^{2\pi} \sqrt{t^2 \times 1 + 4t^2}\, dt.$$

$$= \int_0^{2\pi} \sqrt{t^2 + 4t^2}\, dt.$$

$$= \int_0^{2\pi} \sqrt{5t^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{5} \times \sqrt{t^2}\, dt$$

$$= \sqrt{5} \int_0^{2\pi} \sqrt{t^2}\, dt$$

$$= \sqrt{5} \int_0^{2\pi} (t^2)^{1/2}\, dt$$

$$= \sqrt{5} \int_0^{2\pi} t\, dt$$

$$= \sqrt{5} \left[\frac{t^2}{2}\right]_0^{2\pi}$$

$$= \sqrt{5} \left[\frac{2\pi^2}{2} - \frac{0^2}{2}\right]$$

$$= \sqrt{5} \cdot \left[\frac{4\pi^2}{2} - 0\right]$$

$$= \sqrt{5} \left[2\pi^2\right]$$

☆ Motion on a curve.

Concider a vector valued function $r(t) = x(t)i + y(t)j + z(t)k$ if $x, y, \&\ z$ are twice differentiable functions of t, then the velocity vector ($v(t)$), acceleration $a(t)$ & speed $\|v(t)\|$ are defined by.

$$v(t) = r'(t) = x'(t)i + y'(t)j + z'(t)k$$

$$a(t) = r''(t) = x''(t)\, i + y''(t)j + z''(t)k$$

$$\text{speed} = \|v(t)\| = \sqrt{(x'(t))^2 + [y'(t)]^2 + (z'(t))^2}$$

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→ The position vector of a particle in space at time t is,

$r(t) = (t+1)i + (t^2-1)j + 2tk$

(a) find the particle's velocity & accelaration vectors.

(b) find the particle's speed & direction of motion at t=1

(c) find the angle between the velocity & accelaration vectors at t=1

⇒ (a) $r(t) = (t+1)i + (t^2-1)j + 2tk$.

velocity $v(t) = r'(t) = 1i + \underline{\underline{2tj + 2k}}$

accelaration $a(t) = r''(t) = \underline{\underline{2j}}$

(b) Speed $\|v(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$

$= \sqrt{1^2 + (2t)^2 + (2)^2}$

$= \sqrt{1 + 4t^2 + 4}$

$= \underline{\underline{\sqrt{5 + 4t^2}}}$

at t=1,

Speed $= \sqrt{5 + 4 \times 1^2}$

$= \sqrt{5+4} = \sqrt{9} = \underline{\underline{3}}$

$$\boxed{\text{Direction} = \frac{v(t)}{\|v(t)\|}}$$

$$\text{Direction} = \frac{v(t)}{\|(v(t))\|} = \frac{i+2tj+2k}{\sqrt{1^2+(2t)^2+2^2}}$$

$$= \frac{i+2tj+2k}{\sqrt{1+4t^2+4}}$$

$$= \frac{i+2tj+2k}{\sqrt{5+4t^2}}$$

at $t=1$, $\text{Direction} = \dfrac{i+(2\times1)j+2k}{\sqrt{5+4\times1^2}}$

$$= \frac{i+2j+2k}{\sqrt{9}} = \frac{i+2j+2k}{3}$$

(c) $\boxed{\text{angle } \cos\theta = \dfrac{v(t)\cdot a(t)}{\|v(t)\|\,\|a(t)\|}}$

$$\cos\theta = \frac{[i+2tj+2k]\cdot[2j]}{\sqrt{5+4t^2} \times \sqrt{(2)^2}}$$

$$= \frac{(1\times0)+(2t\times2)+(2\times0)}{\sqrt{5+4t^2} \times \sqrt{4}} = \frac{4t}{2\sqrt{5+4t^2}}$$

at $t=1$,

$$\cos\theta = \frac{4\times1}{2\sqrt{5\times4(1)^2}} = \frac{4}{2\sqrt{5+4}} = \frac{4}{2\sqrt{9}} = \frac{4}{2\times3} = \frac{4}{6}$$

→ A person on a hang glider is spuraling upward due to hapidly rising air on a path having position vector.

$r(t) = 3\cos t\, i + 3\sin t\, j + t^2 k$, $0 \le t \le 4\pi$.

Then evaluate the following:

(a) The velocity & accelaration vectors.

(b) The glider's speed at any time 't'.

(c) The time, when the gliders accelaration is orthogonal to its velocity.

$\Rightarrow$ (a) $= r(t) = 3\cos t\, i + 3\sin t\, j + t^2 k$

velocity $= r'(t) = -3\sin t\, i + 3\cos t j + 2k$

(b) speed $= \|v(t)\| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (2t)^2}$

$= \sqrt{9\sin^2 t + 9\cos^2 t + 4t^2}$

$= \sqrt{9[\sin^2 t + \cos^2 t]\ 4t^2}$

$= \sqrt{9 \times 1 + 4t^2}$

$= \sqrt{9 + 4t^2}$

(c) Given accelaration & velocity is orthogonal.

ie; $v(t) \cdot a(t) = 0$

$\Rightarrow [(-3\sin t\, i) + 3\cos t j + 2tk] \cdot [-3\cos t\, i - 3\sin t j + 2k] = 0$

$\Rightarrow 9 \sin t \cos t - 9 \sin t \cos t + 4t = 0$

$\Rightarrow 4t = 0$

$\Rightarrow \underline{t = 0}$

If a particle moves with a constant speed, show that its velocity vector v is $\perp^r$ to its accelaration vector?

$\Rightarrow$ Suppose a particle moves with constant speed. Then speed = constant.

$$\Rightarrow \| v(t) \| = \text{constant} \ \forall t$$

$\therefore v(t) \cdot v(t) = \| v(t) \|^2 = c$, a constant.

$[\because \|v(t)\| = \text{constant}]$

Differentiating both sides of the above eq^n with respect to t, we obtain.

$$\frac{d}{dt}[v(t) \cdot v(t)] = \frac{d}{dt}(c).$$

$$\frac{d}{dt}[v(t) \cdot v(t) = 0. \quad [\because \tfrac{d}{dt}\text{constant} = 0]$$

then by product rule,

$$v(t) \cdot \frac{d}{dt} v(t) + v(t) \cdot \frac{d}{dt} v(t) = 0$$

$$2\, v(t) \cdot \frac{d}{dt} v(t) = 0$$

$$v(t) \cdot \frac{d}{dt}(v(t)) = \frac{0}{2}$$

$$= 0 //$$

$v(t) \cdot a(t) = 0 \quad \forall \ t. \quad \left[\because \frac{d}{dt} v(t) = a(t)\right]$

Hence v is $\perp^r$ to a.