Problem 4. *Determine whether the binary operation $*$ defined below is commutative and whether is associative :*

(i) $*$ *defined on* $\mathbb{Q}$ *by letting* $a * b = ab + 1$.

(ii) $*$ *defined on* $\mathbb{Z}^+$ *by letting* $a * b = a^b$.

Solution. (i) For any $a, b \in \mathbb{Q}$,

$$a * b = ab + 1 \quad \text{and} \quad b * a = ba + 1 = ab + 1.$$

Therfore $a*b=b*a$ for all $a, b\in \mathbb{Q}$ and so $*$ is commutative on $\mathbb{Q}$.

For any $a, b, c\in \mathbb{Q}$,

$$(a*b)*c=(ab+1)*c=(ab+1)c+1=abc+c+1$$

and $$a*(b*c)=a*(bc+1)=a(bc+1)+1=abc+a+1.$$

Hence if $a\neq c$, then $(a*b)*c\neq a*(b*c)$. Therefore $*$ is not associative on $\mathbb{Q}$.

(ii) For $2, 3\in \mathbb{Z}^+$, $2*3=2^3=8$ and $3*2=3^2=9$.

Therefore $2*3\neq 3*2$. Hence $*$ is not commutative on $\mathbb{Z}^+$.

For $2, 3, 4\in \mathbb{Z}^+$, $(2*3)*4=(2^3)*4=(2^3)^4=2^{12}$

and $$2*(3*4)=2*(3^4)=2*81=2^{81}.$$

Therefore $(2*3)*4\neq 2*(3*4)$. Hence $*$ is not associative on $\mathbb{Z}^+$.

Problem 5. *Prove that the operation $\oplus$ on $\mathbb{Z}$, defined by*

$$m\oplus n=mn-m-n+2$$

is a binary operation with identity element.

Solution. For any two integers m and n, $m\oplus n=mn-m-n+2$ is an integer. Hence $m\oplus n\in \mathbb{Z}$, for all $m, n\in \mathbb{Z}$. Therefore $\oplus$ is a binary operation. For any $m\in \mathbb{Z}$ and for some $x\in \mathbb{Z}$,

$$m\oplus x=m \Rightarrow mx-m-x+2=m \Rightarrow m(x-2)-(x-2)=0$$
$$\Rightarrow (m-1)(x-2)=0 \Rightarrow x=2$$

and $$x\oplus m=m \Rightarrow xm-x-m+2=m \Rightarrow m(x-2)-(x-2)=0$$
$$\Rightarrow (m-1)(x-2)=0 \Rightarrow x=2.$$

Therefore $m\oplus 2=m=2\oplus m$, for all $m\in \mathbb{Z}$ and $2\in \mathbb{Z}$. Hence $\oplus$ is a binary operation with identity on $\mathbb{Z}$ and 2 is the identity element.