Q show that the set

$$n = \{ \ldots -4m, -3m, -2m, -m, 0, m, 2m, 3m, 4m \ldots \}$$

of multiples of integers by a fixed integer m is an abelian group w.r. to addition.

An:-

<u>closure</u> <u>property</u>

let $a, b \in n$.

$a = rm$
$b = sm$.

$a+b = rm + sm$
$= (r+s)m \in n$.

$\therefore$ n is closed under $+$

<u>n_1:</u> [associativity]

$\forall\ a, b, c \in n$.

let $a = rm$
$b = sm$
$c = tm$.

$a+(b+c) = rm + (sm+tm) = (r+s+t)m$

$(a+b)+c = (rm+sm)+tm = (r+s+t)m.$

n_2 [existence of identity]

Let $a \in n$ then $\exists\, 0 \in n$ s.

~~$0 \in n$~~

$a + 0 = a = 0 + a \quad \forall a \in n.$

$\therefore$ identity element exist

n_3 [existence of inverse]

For any $a \in n$

$a = rm.$

$-a = -rm$

$= (-r)m \in n.$

s. $a + (-a) = rm + (-r)m$

$= 0$

$\therefore$ inverse exist.

$\therefore (n, +)$ is a group.

Now we want to show that $(n, +)$ is an abelian group:

Commutative Property

$\forall\, a, b \in n.$

$a + b = b + a.$

Let $a = rm$ $\quad$ $a + b = rm + sm = (r+s)m$

$b = sm$ $\quad$ $b + a = sm + rm = (s+r)m = (r+s)m.$

$\therefore a + b = b + a$

$\therefore$ $(\mathbb{Q}, +)$ is an abelian group.

Q Prove that the set $Q_0 = Q - \{0\}$ of all non-zero rational numbers form an abelian group under multiplication

an:-

closure property

Let $a, b \in Q_0$

$a = \frac{1}{x}$

$b = \frac{1}{y}$

$a \cdot b = \frac{1}{x} \cdot \frac{1}{y}$

$= \frac{1}{xy} \in Q$.

G_1: Associativity

Let $a, b, c \in Q_0$ Let $a = \frac{1}{x}$ $b = \frac{1}{y}$ $c = \frac{1}{2}$

$a * (b * c) = \frac{1}{x} * \left[\frac{1}{y} * \frac{1}{2}\right]$

$= \left[\frac{1}{x} * \frac{1}{y}\right] * \frac{1}{2}$

$= [a * b] * c$.

$\therefore$ Associativity satisfied.

G_2 : Existence of identity

The rational number $a \in G$.

$$a * e = a$$

$$\implies \frac{1}{x} \times \frac{e}{1} = a.$$

$$\cancel{\implies \frac{e}{x} = \frac{a}{1}}$$

$$\cancel{\implies e = ax}$$

$$\cancel{= a.}$$

$$\implies \frac{1}{x} \times e = \frac{1}{x}.$$

$$\implies \frac{e}{x} = \frac{1}{x}.$$

$$\implies e = \frac{x}{x}$$

$$= 1$$

$\therefore$ $1 \in Q_0$

$\therefore$ identity element exist.

G_3 : Existence of inverse

Let $a \in Q_0$ we want to find $a^{-1} \in G$ s.

$$a * a^{-1} = e = a^{-1} * a.$$

$\bar{a}' \neq$

Let $a \in Q_0$ $\frac{1}{a} \in Q_0$ s.

$$a \times \frac{1}{a} = 1$$

$\therefore$ inverse exist.

$\therefore (Q_0, \cdot)$ is a group.

<u>commutative</u>

Let $a, b \in Q_0$

$$a \cdot b = \frac{1}{x} \cdot \frac{1}{y}$$

$$= \frac{1}{y} \cdot \frac{1}{x}$$

$$= b \cdot a.$$

$\therefore$ commutative property satisfied

$\therefore (Q_0, \cdot)$ is a abelian group.

Q prove that the set

$n = \{ \ldots 2^{-4}, 2^{-3}, 2^{-3}, 2^{-2}, 2^{-1}, 1 = 2^0, 2, 2^2 \ldots \}$

is an abelian group w.r. to multiplication.

a:- closure property

Let $a, b \in n$.

$a = 2^m$

$b = 2^n$

$a \cdot b = 2^m \cdot 2^n$

$= 2^{m+n} \in n$ [$m+n = 0$]

$= 2^0 \in n$

$\therefore n$ is closed.

ii) [Associativity]

Let $a, b, c \in n$. $a = 2^m$ $b = 2^n$ $c = 2^p$

~~(a+b)~~

$(a \cdot b) \cdot c = (2^m \cdot 2^n) \cdot 2^p$

$= 2^m \cdot (2^n \cdot 2^p)$

$= a \cdot (b \cdot c)$

$\therefore$ Associative property satisfied.

G_2: Existence of identity element

Let $a \in G$, we want find $e \in G$ s.

$a * e = a = e * a$.

Let $a = 2^m$.

$a * e = e * a$

$\Rightarrow 2^m \cdot e = 2^m$.

$\Rightarrow e = \dfrac{2^m}{2^m}$

$= 1$

$2^0 = 1 = e \in G$.

$\therefore$ 1 is the identity element of G.

$\therefore$ identity element exist.

G_3: Existence of inverse.

Let $a \in G$ we want to find $a^{-1} \in G$ s.

$a * a^{-1} = e$.

Let $a \in G$. $a = 2^m$

$2^m \in G$.

$\dfrac{1}{2^m}$ $2^{-m} \in G$ s.

$2^m \cdot \dfrac{1}{2^m} = 1$

$\therefore \frac{1}{2^m}$ is the inverse of n.

$\therefore$ inverse exist. —— ①

$\Rightarrow (n, \cdot)$ is a group.

<u>commutative ppty</u>

Let $a, b \in n$. Let $a = 2^m$ $b = 2^n$

$a * b = 2^m \cdot 2^n$

$= 2^n \cdot 2^m$

$= b \cdot a$.

$\therefore$ commutative property satisfied. —— ②

From ① & ② we get $(n, \cdot)$ is an abelian group.

Q Show that the set of all positive rational numbers forms an abelian group under the operation defined by $a * b = \frac{ab}{2}$.

An:- Let Q^+ denote the set of all rational number.

<u>Closure property</u>

we have $a * b = \frac{ab}{2}$

$\frac{ab}{2} \in Q^+$

$\therefore$ closure property satisfied.

n_1: Associative

Let $a, b, c \in Q^+$

$a * (b * c)$

$= a * \left[\frac{bc}{2}\right]$

$= \frac{a(bc/2)}{2}$

$= \frac{abc}{4}$ ——— ①

$(a * b) * c$

$= \left[\frac{ab}{2}\right] * c$

$= \frac{\frac{ab}{2} \times c}{2}$

$= \frac{abc}{4}$ ——— ②

From ① & ② we get

$a * (b * c) = (a * b) * c$

∴ Associative property satisfied.

n_2: [Existence of identity element]

Let $a \in Q^+$ we want to find $e \in Q^+$ s.

$a * e = a = e * a$.

Let

$a * e = a$

$\Rightarrow \frac{ae}{2} = \frac{a}{1}$

$\Rightarrow ae = 2a$

$e = \frac{2\cancel{a}}{\cancel{a}}$

$= 2$

$\therefore$ we get

$a * 2 = a$.

since $2 \in Q^+$

2 is the identity element

$\therefore$ identity element exist.

n_3: [existence of inverse]

Let $a \in Q^+$ we want to find $a^{-1} \in Q^+$ s.

$a * a^{-1} = e = a^{-1} * a$.

$a * a^{-1} = e$

$\Rightarrow \frac{a a^{-1}}{2} = 2$.

$\Rightarrow a a^{-1} = 4$

$a^{-1} = \frac{4}{a}$.

$a \in Q^+$ and $4 \in Q^+$ $\therefore \frac{4}{a} \in Q^+$

$\frac{4}{a}$ is the inverse of the set.

$\therefore$ $a * a^{-1} = e = a^{-1} * a$

$\therefore$ inverse exist.

<u>commutative property</u>

Let $a, b \in Q^+$

we want to S.T $a*b = b*a$.

$$a*b = \frac{ab}{2} \quad \text{——— (1)}$$

$$b*a = \frac{ba}{2}$$
$$= \frac{ab}{2} \quad \text{——— (2)}$$

From (1) & (2) we get $*$ is commutative.

$$a*b = b*a.$$

$\therefore$ $(Q^+, *)$ is an abelian group.