§ 2. ALGEBRAIC STRUCTURE

A non-empty set S together with one or more binary operations defined on S is called an an ***algebraic structure***. Suppose '$*$' and '$\circ$' are two binary operations defined on S. Then $(S, *, \circ)$ is an algebraic structure.

A non-empty set S together with a binary operation operation $*$ on S, denoted by $(S, *)$ or $\langle S, * \rangle$, is called a ***binary structure*** or ***binary algebraic structure***.

Definition. *Let* $(S, *)$ *and* $(S', *')$ *be two binary algebraic structures. An* ***isomorphism of*** *S with* S' *is a one-to-one function mapping S onto* S' *such that*

$$\phi(x * y) = \phi(x) *' \phi(y), \quad \text{for all } x, y \in S. \qquad \ldots (A)$$

If such a map ϕ *exists, then S and* S' ***are isomorphic binary structures****, which we denote by* $S \simeq S'$.

The property of the isomorphism ϕ, displayed by equation (A) is usually known as ***homomorphism property***. Thus a one-to-one onto map satisfying the homomorphism property is an isomorphism.

Remark. To show that two binary structures $(S, *)$ and $(S', *')$ are isomorphic we have to proceed as follows :

Step 1. *Define the function* ϕ *that gives the isomorphism of S with* S'. This means that we have to describe in some fashion, what $\phi(s)$

is to be for every $s \in S$.

<u>Step 2.</u> *Show that ϕ is a one-to-one function.* That is, we have to show that for any $x, y \in S$,

$$\phi(x) = \phi(y) \text{ in } S' \Rightarrow x = y.$$

<u>Step 3.</u> *Show that ϕ is onto S'.* That is, we have to show that for any $s' \in S'$, there exists $s \in S$ such that $\phi(s) = s'$.

<u>Step 4.</u> *Show that* $\phi(x * y) = \phi(x) *' \phi(y)$, *for all* $x, y \in S$.

Problem 6. *Show that the binary structure* $(\mathbb{R}, +)$, *where* $+$ *is the usual addition, is isomorphic to the binary structure* $(\mathbb{R}^+, \cdot)$, *where* $\cdot$ *is the usual multiplication and* $\mathbb{R}^+$ *is the set of all positive real numbers.*

Solution. Define $\phi : \mathbb{R} \to \mathbb{R}^+$ by

$$\phi(x) = e^x, \quad \text{for all } x \in \mathbb{R}.$$

Then ϕ is a well defined map from $\mathbb{R}$ to $\mathbb{R}^+$. Also for any $x, y \in \mathbb{R}$,

$$\phi(x) = \phi(y) \Rightarrow e^x = e^y \Rightarrow \ln e^x = \ln e^y \Rightarrow x = y.$$

Therefore ϕ is one-to-one.

Choose any $r \in \mathbb{R}^+$. Then $\ln r \in \mathbb{R}$ and $\phi(\ln r) = e^{\ln r} = r$. Hence ϕ is onto $\mathbb{R}^+$. Thus ϕ is a one-to-one onto map.

For any $x, y \in \mathbb{R}$,

$$\phi(x + y) = e^{x+y} = e^x \cdot e^y = \phi(x) \cdot \phi(y).$$

Therefore ϕ satisfies the homomorphism property and hence is an isomorphism. Since $\phi : \mathbb{R} \to \mathbb{R}^+$ is an isomorphism, the binary structures $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$ are isomorphic.

Problem 7. *Let* $2\mathbb{Z} = \{2n : n \in \mathbb{Z}\}$, *so that* $2\mathbb{Z}$ *is the set of all even integers, positive, negative and zero. Prove that* $(\mathbb{Z}, +)$ *is isomorphic to* $(2\mathbb{Z}, +)$, *where* $+$ *is the usual addition.*

Solution. Define $\psi : \mathbb{Z} \to 2\mathbb{Z}$ by

$$\psi(n) = 2n, \quad \text{for all } n \in \mathbb{Z}.$$

Then ψ is a well defined map from $\mathbb{Z}$ to $2\mathbb{Z}$. Also for any $n, m \in \mathbb{Z}$,

$$\psi(n) = \psi(m) \Rightarrow 2n = 2m \Rightarrow n = m.$$

Therefore ψ is one-to-one.

Choose any $n \in 2\mathbb{Z}$. Then $n = 2m$ for $m = n/2 \in \mathbb{Z}$ and so

$$\psi(m) = 2m = 2(n/2) = n.$$

Hence ϕ is onto $2\mathbb{Z}$. Thus ϕ is a one-to-one onto map.

For any $n, m \in \mathbb{R}$,

$$\psi(n+m) = 2(n+m) = 2n + 2m = \psi(n) + \psi(m).$$

Therefore ψ satisfies the homomorphism property and hence is an isomorphism. Since $\psi : \mathbb{Z} \to 2\mathbb{Z}$ is an isomorphism, the binary structures $(\mathbb{Z}, +)$ and $(2\mathbb{Z}, +)$ are isomorphic.