characterization of intervals

characterization theorem

If S is a subset of $\mathbb{R}$ that contains at least 2 points and has the property

$$x, y \in S \text{ and } x < y \Longrightarrow [x, y] \subseteq S$$

then S is an interval.

proof

Here we have to consider 4 cases

(i) S is bounded.

(ii) S is bounded above but not below.

(iii) S is bounded below but not above.

(iv) S is neither bounded above nor below.

Case-1 S is bounded.

Let $a = \inf(S)$

$b = \sup(S)$

Then $S \subseteq [a, b]$

suppose $a, b \notin S$

then $S \subseteq (a, b)$ —— ①

Now we want to show that $(a, b) \subseteq S$

Let $z \in (a, b)$

$\Longrightarrow a < z < b$

$\Longrightarrow$ 'z' is not a lowerbound of S [$\because a = \inf S$

$\therefore \exists x \in S$ s.t. $x < z$ —— ②

Also z is not a upperbound of S

$\exists y \in S$ s.t. $z < y$ —— ③

compare ② & ③ we get

$x < z < y$

$\Longrightarrow z \in (x, y)$

by theorem $x < y$

$\Longrightarrow z \in [x, y] \subseteq S$

$\Longrightarrow z \in S$.

$\therefore$ we get

$(a, b) \subseteq S$ ——— (4)

From (1) and (4) we get

$S = (a, b)$

If $a \in S$ and $b \in S$

then $S = [a, b]$

Case-2

['S' is bounded above but not below]

Let $b = \sup(S)$

then $S \subseteq [-\infty, b]$

If $b \notin S$

$\Longrightarrow S \subseteq (-\infty, b)$ ——— (1)

we want to show that

$(-\infty, b) \subseteq S$

Let $z \in (-\infty, b)$

$\Longrightarrow -\infty < z < b$

$\Longrightarrow z < b$

$\Longrightarrow$ z is not an upper bound of S

$\exists\, y \in S$ s.t. $z < y$ ——— (2)

also 's' is not bounded below we can find $x \in S$ $\ni$

$x < z$ ——— ③

from ④ & ③

$x < z < y$

$\Longrightarrow z \in (x, y)$

$\Longrightarrow z \in [x, y) \subseteq S$

$\Longrightarrow z \in S$

$\Longrightarrow (-\infty, b) \subseteq S$

If $b \in S$ then $S = (-\infty, b]$

<u>case-3</u>

's' is bounded below but not bounded above.

let $a = \inf(S)$

$S \subseteq [a, \infty)$

If $a \notin S$

we want to S.T $(a, \infty) \subseteq S$

let $z \in (a, \infty) \Longrightarrow a < z < \infty$

$\Longrightarrow a < z$

$\Longrightarrow z$ is not a lower bound of S

$\exists x \in S \ni x < z$ ——— ④

also z is not bounded above

$\exists y \in S \ni$ ~~y~~

~~y~~ $z < y$ ——— ⑤

from ④ & ⑤ we get

$x < z < y$

$\Longrightarrow z \in (x, y) \Longrightarrow z \in [x, y) \subseteq S$

$\Longrightarrow z \in S$

$\therefore [a, \infty) \subseteq S$

$\therefore S = [a, \infty)$

Case IV

S is neither bounded below nor above

For any real number z, we can find $x, y \in S$

S. $x < z < y$

But hypothesis $x, y \in S$ and

$$x < y \Rightarrow [x, y] \subseteq S$$

$\therefore z \in [x, y] \subseteq S$

ie, $z \in S$

Theorem (Nested intervals property)

Let $I_n = [a_n, b_n]$ $n \in N$ is a nested sequence of closed - bounded intervals, then there exist a number $\xi \in R$ s: $\xi \in I_n$ $\forall n \in N$ furthermore if lengths $b_n - a_n$ of I_n satisfy

$$\inf \{ b_n - a_n : n \in N \} = 0$$

~~Proof~~ then the common element ξ is unique.

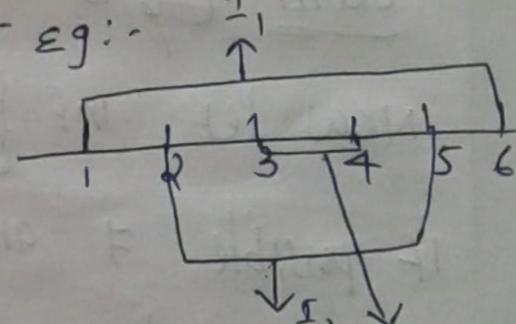
Proof

By hypothesis $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots$

$\therefore [a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots \supseteq [a_n, b_n] \supseteq [a_{n+1}, b_{n+1}] \supseteq \cdots$

and so $a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \cdots \leq b_{n+1} \leq b_n \cdots \leq b_2 \leq b_1$ —— (1)

Eg:-



$I_1 = [1, 6]$
$I_2 = [2, 5]$ ~~1<2<3<4<5<6~~
$I_3 = [3, 4]$ ~~$a_1 < a_2 < a_3 < b_3$~~

$1 < 2 < 3 < 4 < 5 < 6$
$a_1 < a_2 < a_3 < b_3 < b_2 < b_1$

Let $A = \{ a_n : n \in N \}$

since $a_n \leq b_1$ $\forall n \in N$

A is a nonempty subset of R. which is bounded above.

Hence by supremum property

Let $\sup A = \xi$

then from definition

$a_n \leq \xi$ $\forall n \in N$ —— (2)

[ξ is an upperbound]

①

$\Longrightarrow \forall n \in N$

$a_k \leq b_n \quad \forall k \in N$

Hence b_n is an upperbound of the set A.

Hence by the definition of supremum

$$\xi \leq b_n \quad [\xi \text{ is the supremum}] \text{ ——— ③}$$

From ② and ③ we get

$$a_n \leq \xi \leq b_n \quad \forall n \in N$$

$\Longrightarrow \xi \in [a_n, b_n]$ for all $n \in N$

<u>claim</u> [ξ is unique]

Now let $\inf \{ b_n - a_n : n \in N \} = 0$

If possible $\exists$ another number η s.t. $\eta \in I_n \ \forall n \in N$

$\xi \in I_n$ and $\eta \in I_n$

$\Longrightarrow \underbrace{a_n \leq \xi \leq b_n}_{①}$ and $\underbrace{a_n \leq \eta \leq b_n}_{②}$

multiply ① by -1

① $\Rightarrow -a_n \geq -\xi \geq -b_n \Longrightarrow -b_n \leq -\xi \leq -a_n$ ——— ③

② $\Rightarrow a_n \leq \eta \leq b_n$ ——— ④

④ + ③

$a_n - b_n \leq \eta - \xi \leq b_n - a_n$

$\Rightarrow -b_n + a_n \leq \eta - \xi \leq b_n - a_n$

$\Rightarrow -(b_n - a_n) \leq \eta - \xi \leq b_n - a_n$

$\Rightarrow |\eta - \xi| \leq b_n - a_n$ $\left[\begin{array}{l} |x| < \varepsilon \\ \Rightarrow -\varepsilon < x < \varepsilon \end{array} \right]$

$\forall n \in \mathbb{N}$

since $\varepsilon > 0$ cannot be a lowerbound of the set.

$\{ b_n - a_n : n \in \mathbb{N} \}$

Hence $\exists$ an integer $m \in \mathbb{N}$ s.t. and we have $\varepsilon > 0$

$b_m - a_m < \varepsilon$

$\therefore 0 \leq b_m - a_m < \varepsilon$

$|\eta - \xi| \leq b_m - a_m < \varepsilon$

ε is arbitrary we get

$|\eta - \xi| = 0$

$\Rightarrow \eta = \xi$

$\Rightarrow \xi$ is unique.

The uncountability of R

Theorem

The set R of real numbers is not countable.

Proof

We shall prove the unit interval $I = [0,1]$ is an uncountable set. This implies that the set R is an uncountable set.

If possible, let I be a countable set. Then we can enumerate the set as

$$I = \{x_1, x_2, x_3 \ldots x_n \ldots\}$$

Let I_1 be a closed subinterval of I s.t. $x_1 \notin I_1$

I_2 be a closed subinterval of I_1 s.t. $x_2 \notin I_2$

and so on.

In this way we obtain

$$I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq I_{n+1} \supseteq \ldots$$

such that $I_n \subseteq I$ and $x_n \notin I_n$ $\forall n$.

Then by nested interval property $\exists$ a real number $\xi \in I$ s.t. $\xi \in I_n$ $\forall n$.

But $\xi \in I_n$

$$\implies \xi \neq x_n \quad [x_n \notin I_n]$$

$\forall n$.

Hence enumeration of I is not a complete listing of the elements of I, as we claime.

this is a contradiction.

$\implies$ I is an uncountable set