CHAPTER - 5

COMPLEX NUMBERS

§ 1. INTRODUCTION

The concept of complex numbers was originated out of the need to solve quadratic equations with no real roots i.e., quadratic equations with discriminant less than zero. In higher secondary classes, you must have defined a complex number to be a number of the form $x + iy$ where x and y are real numbers, $i = \sqrt{-1}$. In this chapter, we are going to introduce a new definition, which will actually establishes a one-one correspondence between the set of all complex numbers and $\mathbb{R}^2$, the Euclidean plane.

Definitions. *A complex number z is defined as an ordered pair (x, y) of real numbers x and y, written*

$$z = (x, y), \qquad x, y \in \mathbb{R}.$$

Here x is called the ***real part*** and y the ***imaginary part*** of z, written

$$x = \text{Re } z \quad \text{and} \quad y = \text{Im } z.$$

For example, Re $(4, -3) = 4$ and Im $(4, -3) = -3$.

§ 2. ALGEBRA OF COMPLEX NUMBERS

Equality of Complex Numbers

Two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are said to be equal if and only if their real parts are equal and their imaginary parts are equal. i. e., $z_1 = z_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

Sum of Complex Numbers

The sum $z_1 + z_2$ of two complex numbers, $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is a complex number given by

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

For example, $(3, -2) + (4, 3) = (7, 1)$.

Product of Complex Numbers

The product $z_1 z_2$ of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$

is a complex number given by

$$z_1z_2 = (x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1).$$

For example, $(3, -2)(4, 3) = (12 + 6, 9 - 8) = (18, 1)$.

Difference of Complex Numbers

The difference $z_1 - z_2$ of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is a complex number given by

$$z_1 - z_2 = (x_1, y_1) - (x_2, y_2) = (x_1 - x_2, y_1 - y_2)$$

For example, $(3, -2) - (4, 3) = (3 - 4, -2 - 3) = (-1, -5)$.

Quotient of Complex Numbers

The quotient of two complex numbers z_1 and z_2 denoted by $\frac{z_1}{z_2}$ is a complex number z such that $z_2z = z_1$.

Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ and $z = (x, y)$.

Then, $z_2z = z_1 \Rightarrow (x_2, y_2)(x, y) = (x_1, y_1)$

$$\Rightarrow (x_2x - y_2y, x_2y + y_2x) = (x_1, y_1)$$

$$\Rightarrow x_2x - y_2y = x_1 \quad \text{and} \quad x_2y + y_2x = y_1.$$

Solving these equations for x and y, we get

$$x = \frac{x_1x_2 + y_1y_2}{x_1^2 + y_1^2} \quad \text{and} \quad y = \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}.$$

$$\therefore \quad \frac{z_1}{z_2} = \frac{(x_1, y_1)}{(x_2, y_2)} = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\right).$$

§ 3. REPRESENTATION IN THE FORM $z = x + iy$

A complex number whose imaginary part is zero i.e., a complex number of the form $(x, 0)$, has the properties of the real number x and hence we identify $(x, 0)$ with the real number x. The complex number whose real part is zero is called a ***pure imaginary number***.

The pure imaginary number $(0, 1)$ is called the ***imaginary unit*** and is usually denoted by i.

From the definition of multiplication,

$$i^2 = (0, 1)(0, 1) = (0 - 1, 0 + 0) = (-1, 0) = -1.$$

§4. ALGEBRAIC PROPERTIES

Various properties of addition and multiplication of complex numbers are the same as for real numbers. We list here the more basic of these algebraic properties and verify some of them.

Commutative laws

Addition and multiplication of complex numbers is cummutative, i.e., if z_1 and z_2 are any two complex numbers,

$$z_1 + z_2 = z_2 + z_1 \text{ and } z_1z_2 = z_2z_1.$$

Proof. Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. Then since addition and multiplication of real numbers is cummutative, we have,

$$z_1 + z_2 = (x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2, y_1 + y_2)$$
$$= (x_2 + x_1, y_2 + y_1) = (x_2, y_2) + (x_1, y_1) = z_2 + z_1$$

and

$$z_1z_2 = (x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)$$
$$= (x_2x_1 - y_2y_1, y_2x_1 + x_2x_1) = (x_2, y_2)(x_1 y_1) = z_1z_2.$$

Associative laws

Addition and multiplication of complex numbers are associative. i.e., if z_1, z_2 and z_3 are any three complex numbers,

$$(z_1+ z_2) + z_3 = z_1+ (z_2+ z_3) \text{ and } (z_1z_2)z_3 = z_1(z_2z_3).$$

Associative laws can also be proved easily as above, using the fact that addition and multiplication of real numbers is associative.

Distributive laws

If z_1, z_2, and z_3 are any three complex numbers,

$$z_1(z_2+ z_3) = z_1 z_2 + z_1 z_3 \text{ and } (z_1 + z_2)z_3 = z_1z_3 + z_2z_3.$$

Distributive laws also follows easily from the definitions of the addition and multiplication of complex numbers and the fact that real numbers obey this law.

Existence of additive and multiplicative identity

The additive identity $0 = (0, 0)$ and the multiplicative identity $1 = (1, 0)$ for real numbers carry over to the entire complex number system.

i.e., $z + 0 = z = z + 0$ and $z \cdot 1 = z = z \cdot 1$ for any complex number z.

Note that 0 and 1 are the only complex numbers with such properties.

Existence of additive inverse

Associate with each complex number $z = (x, y)$, there exists a complex number, denoted by $-z$ and defined by $-z = (-x, -y)$, which satisfies the equation $z + (-z) = (x, y) + (-x, -y) = (0, 0) = 0$. This complex number $-z$ is known as the ***additive inverse*** of z.

Note that, for a given complex number, there is only one additive inverse, since the equation $(x, y) + (u, v) = (0, 0)$ implies that $u = -x$ and $v = -y$.

Existence of multiplicative inverse

For any non-zero complex number $z = (x, y)$, there is a number z^{-1}, known as ***multiplicative inverse*** of z, such that $zz^{-1} = 1$

Thus multiplicative inverse is less obvious than the additive inverse.

To find it, let $z^{-1} = (u, v)$.

Then,
$$\begin{aligned} zz^{-1} = 1 &\Rightarrow (x, y)(u, v) = 1 \\ &\Rightarrow (xu - yv, xv + yu) = (1, 0) \\ &\Rightarrow xu - yv = 1 \text{ and } yu + xv = 0. \end{aligned}$$

Since $z = (x, y) \neq (0,0)$, the above system of linear equations in u and v, will have a unique solution, given by

$$u = \frac{x}{x^2 + y^2} \quad \text{and} \quad v = \frac{-y}{x^2 + y^2}.$$

Hence multiplicative inverse of z is given by

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right).$$

Remark 1. The existence of multiplicative inverses enables us to show that if a product $z_1 z_2$ is zero, then so is at least one of the factors z_1 and z_2.

i. e.,
$$z_1 z_2 = 0 \Rightarrow z_1 = 0 \text{ or } z_2 = 0.$$

§ 6. COMPLEX CONJUGATES

The ***complex conjugate*** or simply the ***conjugate***, of a complex number

$$z = (x, y) = x + i\,y$$

is defined as the complex number

$$\bar{z} = (x, -y) = x - iy.$$

The point $\bar{z}$ is the reflection of the point z in the x- axis, as shown in the figure.

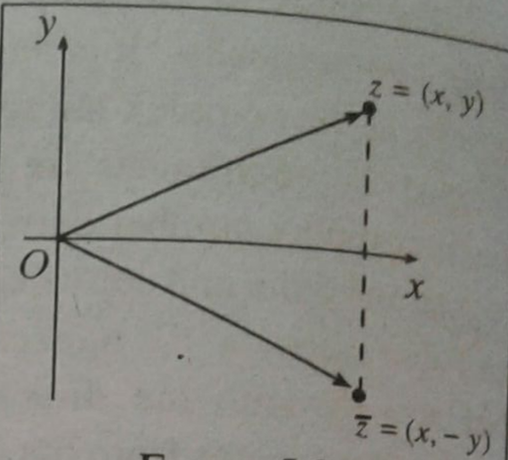


FIGURE 5.3

Properties of complex conjugates

If z_1 and z_2 are any two complex numbers, then

(i) $\dfrac{z_1 + \overline{z_1}}{2} = \operatorname{Re} z_1$; $\dfrac{z_1 - \overline{z_1}}{2i} = \operatorname{Im} z_1$. (ii) $\overline{\left(\overline{z_1}\right)} = z_1$.

(iii) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$. (iv) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$.

(v) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$. (vi) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$.

Proof. Let $z_1 = (x_1, y_1) = x_1 + i\,y_1$ and $z_2 = (x_2, y_2) = x_2 + iy_2$.
Then $\overline{z_1} = (x_1, -y_1) = x_1 - iy_1$ and $\overline{z_2} = (x_2, -y_2) = x_2 - iy_2$.

(i) $$\frac{z_1 + \overline{z_1}}{2} = \frac{x_1 + iy_1 + x_1 - iy_1}{2} = x_1 = \operatorname{Re} z_1$$

and $$\frac{z_1 - \overline{z_1}}{2i} = \frac{x_1 + iy_1 - (x_1 - iy_1)}{2i} = y_1 = \operatorname{Im} z_1.$$

(ii) $$\overline{\left(\overline{z_1}\right)} = \overline{x_1 - iy_1} = \overline{x_1 + i(-y_1)} = x_1 - i(-y_1) = x_1 + iy_1 = z_1.$$

(iii) $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$.

$\therefore \quad \overline{z_1 + z_2} = (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2)$

$= \overline{z_1} + \overline{z_2}$.

The other properties can be similarly proved and they are left as exercise to the students.

§7. MODULUS OF A COMPLEX NUMBER

The ***modulus*** or ***absolute value***, of a complex number $z = x + iy$ is defined as the non-negative real number $\sqrt{x^2 + y^2}$ and is denoted by $|z|$.

i. e.,
$$|z| = \sqrt{x^2 + y^2}.$$

For any complex number $z = x + iy$, $\bar{z} = x - iy$ and hence
$$z\bar{z} = x^2 + y^2.$$

Hence modulus of z denoted by $|z|$, can also be defined as the non-negative square root of $z\bar{z}$.

i. e.,
$$|z| = \sqrt{z\bar{z}}.$$

Geometrically, the number $|z|$ is the distance between the point $z = (x, y)$ and the origin. Consequently, $|z_1 - z_2|$ is the distance between the points z_1 and z_2. This fact is also evident from the definition, since
$$|z_1 - z_2| = |(x_1 - x_2) + i(y_1 - y_2)| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Remark. 1 The statement $|z_1| > |z_2|$ means that the point z_1 is farther from the origin than is the point z_2. Thus elementary notion of linear order, greater than or less than, is applicable to absolute values because they are real numbers. However the statement $z_1 > z_2$ or $z_1 < z_2$ are meaningless, unless z_1 and z_2 are both real.

Remark 2. Associated with each complex number z, there are three real numbers Re z, Im z and $|z|$. They are related by the equation
$$|z|^2 = (\text{Re}\, z)^2 + (\text{Im}\, z)^2$$

and the conditions $|z| \geq |\text{Re}\, z| \geq \text{Re}\, z$ and $|z| \geq |\text{Im}\, z| \geq \text{Im}\, z$.

Properties of Moduli of Complex Numbers

If z_1 and z_2 are any two complex numbers, then

(i) $|\overline{z_1}| = |z_1|$.

(ii) $|z_1 z_2| = |z_1| \cdot |z_2|$.

(iii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

(iv) $|z_1 + z_2| \le |z_1| + |z_2|$

(v) $|z_1 + z_2| \ge |z_1| - |z_2|$

(vi) $|z_1 - z_2| \ge ||z_1| - |z_2||$.

Proof. Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$.

(i) $|z_1| = \sqrt{x_1^2 + y_1^2}$ and $\overline{z_1} = (x_1, -y_1) = x_1 - iy_1$.

$$\therefore |\overline{z_1}| = |x_1 + i(-y_1)| = \sqrt{x_1^2 + (-y_1)^2} = \sqrt{x_1^2 + y_1^2} = |z_1|.$$

(ii) $$|z_1 z_2|^2 = (z_1 z_2) \cdot \overline{(z_1 z_2)} = (z_1 z_2) \cdot (\overline{z_1} \cdot \overline{z_2}) \qquad [\because |z|^2 = z\bar{z}]$$

$$= (z_1 \overline{z_1}) \cdot (z_2 \overline{z_2}) = |z_1|^2 \cdot |z_2|^2.$$

Taking square roots of both sides, we get

$$|z_1 z_2| = |z_1| \cdot |z_2|.$$

(iii) $$\left|\frac{z_1}{z_2}\right|^2 = \left(\frac{z_1}{z_2}\right) \cdot \overline{\left(\frac{z_1}{z_2}\right)} \qquad [\because |z|^2 = z\bar{z}]$$

$$= \frac{z_1}{z_2} \cdot \frac{\overline{z_1}}{\overline{z_2}} = \frac{z_1 \overline{z_1}}{z_2 \overline{z_2}} = \frac{|z_1|^2}{|z_2|^2}.$$

Taking square roots on both sides, we get

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

(iv) $$|z_1 + z_2|^2 = (z_1 + z_2) \cdot \overline{(z_1 + z_2)} \qquad [\because |z|^2 = z\bar{z}]$$

$$= (z_1 + z_2)(\overline{z_1} + \overline{z_2})$$

$$= z_1 \overline{z_1} + z_1 \overline{z_2} + \overline{z_1} z_2 + z_2 \overline{z_2}$$

$$= |z_1|^2 + z_1 \overline{z_2} + \overline{\overline{z_1 \overline{z_2}}} + |z_2|^2$$

$$= |z_1|^2 + 2\,\mathrm{Re}\left(z_1\overline{z_2}\right) + |z_2|^2$$

$$\le |z_1|^2 + 2\left|z_1\overline{z_2}\right| + |z_2|^2 \qquad \left[\because \mathrm{Re}\, z_1\overline{z_2} \le \left|z_1\overline{z_2}\right|\right]$$

$$\le |z_1|^2 + 2|z_1|\left|\overline{z_2}\right| + |z_2|^2$$

$$= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 = \left[|z_2| + |z_2|\right]^2.$$

Taking square roots of both sides, we get

$$|z_1 + z_2| \le |z_1| + |z_2|.$$

Extension of the above rule: Replacing z_2 in the above result by z_3, we get

$$|z_1 + z_2 + z_3| \le |z_1| + |z_2 + z_3| \le |z_1| + |z_2| + |z_3|.$$

Generalizing, we have

$$|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|.$$

(v) $|z_1| = |(z_1 + z_2) + (-z_2)|$

$$\le |z_1 + z_2| + |-z_2| \qquad \text{[using inequality (iv)]}$$

$$= |z_1 + z_2| + |z_2|. \qquad [\because |-z_2| = |z_2|]$$

$\therefore\ |z_1| - |z_2| \le |z_1 + z_2|$ i. e., $|z_1 + z_2| \ge |z_1| - |z_2|$.

(vi) $|z_1 - z_2|^2 = (z_1 - z_2)\overline{(z_1 - z_2)}$ $\qquad \left[\because |z|^2 = z\bar{z}\right]$

$$= (z_1 - z_2)\left(\overline{z_1} - \overline{z_2}\right) = z_1\overline{z_1} - z_1\overline{z_2} - \overline{z_1}z_2 + z_2\overline{z_2}$$

$$= |z_1|^2 - \left[z_1\overline{z_2} + \overline{z_1\overline{z_2}}\right] + |z_2|^2$$

$$= |z_1|^2 - 2\,\mathrm{Re}\left(z_1\overline{z_2}\right) + |z_2|^2 \qquad \ldots(1)$$

$$\mathrm{Re}\left(z_1\overline{z_2}\right) \le \left|z_1\overline{z_2}\right| = |z_1||z_2| \Rightarrow -2\,\mathrm{Re}\left(z_1\overline{z_2}\right) \ge -2|z_1||z_2|.$$

Hence (1) becomes,

$$|z_1 - z_2|^2 \ge |z_1|^2 - 2|z_1||z_2| + |z_2|^2 = \left[|z_1| - |z_2|\right]^2.$$

Taking square roots, we get

$$|z_1 - z_2| \ge \left||z_1| - |z_2|\right|.$$

Remark 1. Inequalities derived in (iv) and (vi) are known as ***triangle inequalities***. They are equivalent to the Geometrical statements that the length of one side of a triangle is less than or equal to sum of the lengths of the other two sides and that the length of one side of a triangle is greater than or equal to the difference of the length of the other two sides.

Remark 2. In the triangle inequality i. e., in $|z_1+z_2| \le |z_1|+|z_2|$ we get equality iff $\left|z_1\overline{z_2}\right| = \text{Re}\left(z_1\overline{z_2}\right)$ i. e., iff $z_1\overline{z_2}$ is purely real and non-negative i.e., iff $z_1\overline{z_2} = a$, where $a \in \mathbb{R}$ and $a > 0$ i. e., if and only if

$$z_1 = \frac{a}{\overline{z}_2} = \frac{a}{|z_2|^2} z_2 = bz_2, \text{ where } b = \frac{a}{|z_2|^2} \in \mathbb{R} \text{ and } b > 0.$$

Problem 1. *Prove that* $\sqrt{2}|z| \ge |\text{Re}\, z| + |\text{Im}\, z|$.

Solution. Let $z = x + iy$.

Then $\text{Re}\, z = x$, $\text{Im}\, z = y$ and $|z| = \sqrt{x^2+y^2}$.

Since square of a real number is always greater than or equal to zero, we have

$$[|x|-|y|]^2 \ge 0.$$

This implies $|x|^2 + |y|^2 - 2|x||y| \ge 0.$

i. e., $x^2 + y^2 \ge 2|x||y|$. $[\because |x|^2 = x, \ \forall x \in \mathbb{R}]$

Now adding $x^2 + y^2 = |x|^2 + |y|^2$, to both sides of the above inequality, we get

$$2\left(x^2+y^2\right) \ge |x|^2 + |y|^2 + 2|x||y| = [|x|+|y|]^2.$$

Taking square root of both sides, the above inequality becomes

$$\sqrt{2}\sqrt{x^2+y^2} \ge |x|+|y| = |\text{Re}\, z| + |\text{Im}\, z|$$

i. e., $\sqrt{2}\,|z| \ge |\text{Re}\, z| + |\text{Im}\, z|$.

Problem 2. *Prove that*

$$|z_1+z_2|^2 + |z_1-z_2|^2 = 2|z_1|^2 + 2|z_2|^2.$$

Interpret the result geometrically.

Solution. $|z_1+z_2|^2+|z_1-z_2|^2$

$$= (z_1+z_2)\overline{(z_1+z_2)}+(z_1-z_2)\overline{(z_1-z_2)}$$

$$= (z_1+z_2)(\overline{z_1}+\overline{z_2})+(z_1-z_2)(\overline{z_1}-\overline{z_2})$$

$$= z_1\overline{z}_1+z_1\overline{z}_2+z_2\overline{z}_1+z_2\overline{z}_2+z_1\overline{z}_1-z_1\overline{z}_2-z_2\overline{z}_1+z_2\overline{z}_2$$

$$= 2z_1\overline{z_1}+2z_2\overline{z_2}=2|z_1|^2+2|z_2|^2.$$

Let $OABC$ be a parallelogram, as shown in the figure 5.4. Let A and C denote the complex numbers z_1 and z_2.

Then $OA = |z_1| = CB$

and $OC = |z_2| = AB.$

Then $OB = |z_1+z_2|$

and $AC = |z_1-z_2|.$

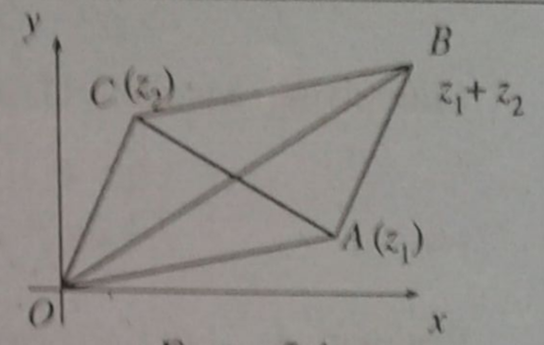


FIGURE 5.4

Hence from the result derived above, it follows that

$$OB^2+AC^2=OA^2+AB^2+BC^2+CO^2.$$

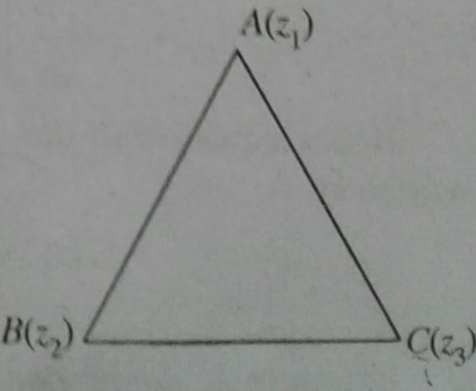
That is, the sum of the squares of the sides of a parallelogram is equal to the sum of the squares of the diagonals.

Remark. Because of this geometric interpretation, above identity is often known as *Parallelogram law.*

Problem 3. *If the complex numbers z_1, z_2, z_3 are the vertices of an equilateral triangle, prove that*

$$z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1.$$

Solution. Let A, B, C be the points z_1, z_2, z_3 respectively. Since the triangle ABC is equilateral $AB = BC = CA.$



i. e., $|z_1-z_2|=|z_2-z_3|=|z_3-z_1|.$

$\therefore\ |z_1-z_2|^2=|z_2-z_3|^2=|z_3-z_1|^2.$

$|z_1-z_2|^2=|z_2-z_3|^2$

$$\Rightarrow (z_1-z_2)(\overline{z}_1-\overline{z}_2)=(z_2-z_3)(\overline{z}_2-\overline{z}_3)$$

$$\Rightarrow \frac{z_1 - z_2}{\bar{z}_2 - \bar{z}_3} = \frac{z_2 - z_3}{\bar{z}_1 - \bar{z}_2}$$

$$\Rightarrow \frac{z_1 - z_2}{\bar{z}_2 - \bar{z}_3} = \frac{z_2 - z_3}{\bar{z}_1 - \bar{z}_2} = \frac{z_1 - z_3}{\bar{z}_1 - \bar{z}_3} \quad \left[\because \frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}\right]$$

$$\Rightarrow \frac{z_1 - z_2}{\bar{z}_2 - \bar{z}_3} = \frac{z_1 - z_3}{\bar{z}_1 - \bar{z}_3} \quad \ldots(1)$$

Also, $|z_2 - z_3|^2 = |z_3 - z_1|^2$

$$\Rightarrow (z_2 - z_3)\left(\overline{z_2} - \overline{z_3}\right) = (z_1 - z_3)\left(\overline{z_1} - \overline{z_3}\right) \quad \ldots(2)$$

Multiplying equations (1) and (2), we get

$$(z_1 - z_2)(z_2 - z_3) = (z_1 - z_3)^2$$

i. e., $\quad z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$

Problem 4. *Find the equation of the circle with centre z_0 and radius r.*

Solution. A circle with centre z_0 and radius r is the collection of all points whose distance from z_0 is r. Let z be any point on the circle. Then equation of the circle with centre z_0 and radius r in the complex plane is

$$|z - z_0| = r.$$

Now $|z - z_0| = r \Rightarrow |z - z_0|^2 = r^2$

$$\Rightarrow (z - z_0)(\bar{z} - \bar{z}_0) = r^2$$

$$\Rightarrow z\bar{z} - z_0\bar{z} - \bar{z}_0 z + z_0\bar{z} = r^2$$

$$\Rightarrow |z|^2 - \left(\bar{z}_0 z + \overline{\bar{z}_0 z}\right) + |z_0|^2 = r^2$$

$$\Rightarrow |z|^2 - 2\,\mathrm{Re}\left(\bar{z}_0 z\right) + |z_0|^2 = r^2.$$

Hence equation of the circle with center z_0 and radius r can also be written as

$$|z|^2 - 2\,\mathrm{Re}\left(\overline{z_0} z\right) + |z_0|^2 = r^2.$$

Example 4. *Show that* $\left|\frac{z+1}{z-1}\right| = 2$ *represents a circle with centre at (5/3, 0) and radius 4/3.*

Solution. $\left|\dfrac{z+1}{z-1}\right|=2 \Rightarrow |z+1|^2=4|z-1|^2$

$$\Rightarrow (z+1)(\bar{z}+1)=4(z-1)(\bar{z}-1)$$

$$\Rightarrow 3z\bar{z}-5z-5\bar{z}+3=0$$

$$\Rightarrow z\bar{z}-\frac{5}{3}z-\frac{5}{3}\bar{z}+1=0$$

$$\Rightarrow \left(z-\frac{5}{3}\right)\left(\bar{z}-\frac{5}{3}\right)-\frac{16}{9}=0$$

$$\Rightarrow \left|z-\frac{5}{3}\right|^2=\frac{16}{9} \Rightarrow \left|z-\frac{5}{3}\right|=\frac{4}{3}.$$

which is a circle with centre $\frac{5}{3}\left[=(\frac{5}{3},0)\right]$ and radius $\frac{4}{3}$.

Aliter. Let $z=x+iy$. Then,

$$\left|\frac{z+1}{z-1}\right|=2 \Rightarrow \frac{|x+iy+1|}{|x+iy-1|}=2 \Rightarrow |x+1+iy|^2=4|x-1+iy|^2$$

$$\Rightarrow (x+1)^2+y^2=4\left[(x-1)^2+y^2\right] \Rightarrow \left(x-\frac{5}{3}\right)^2+y^2=\frac{16}{9}.$$

which is a circle with centre at $\left(\frac{5}{3},0\right)$ and radius $\frac{4}{3}$.