1) The curvature of a Plane curve

Let C be a smooth curve defined by $r(s)$, where $s(t)$ is the arc length of the Parameter. Then the curvature of C at s is

$$\boxed{K = \frac{\|T'(t)\|}{\|r'(t)\|}} \quad , \text{where } T(t) = \frac{r'(t)}{\|r'(t)\|}$$

2) Find the curvature of a circle $a\cos t\, i + a\sin t\, j$ of radius a?

$\Rightarrow r(t) = a\cos t\, i + a\sin t\, j$

$r'(t) = -a\sin t\, i + a\cos t\, j$

$\|r'(t)\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2} = \sqrt{a^2\sin^2 t + a^2\cos^2 t}$

$\|r'(t)\| = \sqrt{a^2(\sin^2 t + \cos^2 t)}$

$\|r'(t)\| = \sqrt{a^2 \times 1} = \sqrt{a^2} = a$

$\therefore T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{-a\sin t\, i + a\cos t\, j}{a} = -\frac{a\sin t\, i}{a} + \frac{a\cos t\, j}{a}$

$T(t) = -\sin t\, i + \cos t\, j$

$T'(t) = -\cos t\, i - \sin t\, j$

$$\|T'(t)\| = \sqrt{(-\cos t)^2 + (-\sin t)^2} = \sqrt{\cos^2 t + \sin^2 t} = \sqrt{1} = 1$$

$\therefore$ curvature $K = \dfrac{\|T'(t)\|}{\|r'(t)\|} = \dfrac{1}{a}$

$\therefore$ The curvature of a circle of radius a is $\dfrac{1}{a}$.

› Find the curvature of the plane curve

$r(t) = t i + (\ln \cos t) j$, $\dfrac{-\pi}{2} < t < \pi/2$.

$\Rightarrow r(t) = t i + (\ln \cos t) j$

$$r'(t) = i + \frac{1}{\cos t} \times -\sin t\, j$$

$$r'(t) = i - \frac{\sin t}{\cos t} j = i - \tan t\, j$$

$$\|r'(t)\| = \sqrt{1^2 + (-\tan t)^2} = \sqrt{1 + \tan^2 t} = \sqrt{\sec^2 t} = \sec t$$

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{i - \tan t\, j}{\sec t}$$

$$= \frac{i}{\sec t} - \frac{\tan t}{\sec t} j$$

$$= \frac{1}{\sec t} i - \frac{\sin t}{\cos t} \times \frac{1}{\sec t} j$$

$$= \cos t\, i - \frac{\sin t}{\cos t} \times \cos t\, j$$

$$T(t) = \cos t\, i - \sin t\, j$$

$$T'(t) = -\sin t\, i - \cos t\, j$$

$$\|T'(t)\| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = \sqrt{\sin^2 t + \cos^2 t} = \sqrt{1} = 1$$

$$\therefore K(t) = \frac{\|T'(t)\|}{\|r'(t)\|} = \frac{\sqrt{1}}{\sec t} = \frac{1}{\sec t} = \underline{\underline{\cos t}}$$

» <u>circle of curvature or osculating circle</u>

The 'circle of curvature' or 'osculating circle' of a curve at a point is the circle, tangent to the curve with the same curvature.

» <u>Radius of curvature</u>

The 'radius of curvature' of the curve at a point P is the radius of the circle of curvature.

$$\text{Radius of curvature } (\rho) = \frac{1}{K}$$

» <u>center of curvature</u>

The 'center of curvature' of curve at a point P is the center of the circle of curvature at P.

> Find the circle of curvature of the Parabola $y = x^2$ at the origin.?

Let $x = t$.

when $x = t$, $y = t^2$ $[\because y = x^2]$.

Hence $r(t) = t\,i + t^2 j$.

$r'(t) = i + 2t\,j$

$\|r'(t)\| = \sqrt{(1)^2 + (2t)^2} = \sqrt{1+4t^2}$

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{i + 2t\,j}{\sqrt{1+4t^2}} = \frac{i}{\sqrt{1+4t^2}} + \frac{2t}{\sqrt{1+4t^2}}\,j$$

$$T(t) = (1+4t^2)^{-1/2}\,i + 2t\,(1+4t^2)^{-1/2}\,j$$

$$T'(t) = -\frac{1}{2}(1+4t^2)^{-3/2} \times 8t\,i + 2\left[t \times -\frac{1}{2}(1+4t^2)^{-3/2} \times 8t + (1+4t^2)^{-1/2}\right] j$$

$$= -4t\,(1+4t^2)^{-3/2}\,i + \left[-8t^2(1+4t^2)^{-3/2} + 2(1+4t^2)^{-1/2}\right] j$$

$$= \frac{-4t}{(1+4t^2)^{3/2}}\,i + \left[\frac{-8t^2}{(1+4t^2)^{3/2}} + \frac{2}{(1+4t^2)^{1/2}}\right] j$$

at the origin, $t = 0$, $\Rightarrow T'(0) = 0\,i + [0 + 2]\,j = 2j$

$r'(0) = i + 2 \times 0\,j = i$

$$\therefore \kappa(0) = \frac{\|T'(0)\|}{\|r'(0)\|} = \frac{\sqrt{(2)^2}}{\sqrt{(1)^2}} = \frac{2}{1} = \underline{\underline{2}}$$

$\therefore$ radius of curvature $\rho = \frac{1}{\kappa} = \underline{\underline{\frac{1}{2}}}$

The center of curvature is $(0, 1/2)$.

The equation of the circle of curvature is

$$(x-0)^2 + (y - 1/2)^2 = \left(\frac{1}{2}\right)^2$$

$$x^2 + y^2 + \frac{1}{4} - y = \frac{1}{4}$$

$$\underline{\underline{x^2 + y^2 - y = 0}}$$

> Find curvature and the radius of curvature at the stated point:

1) $r(t) = 5\cos t\, \hat{i} + 5\sin t\, \hat{j} + t\, k$; $t = \pi/2$

2) $r(t) = e^{t}\, \hat{i} + e^{-t}\, \hat{j} + 2t\, k$; $t = 0$

3) $x = e^{t}\cos t$, $y = e^{t}\sin t$, $z = e^{t}$; $t = 0$

4) $x = \sin t$, $y = \cos t$, $z = \frac{1}{3}t^{3}$; $t = 0$.

>> Unit Tangent Vector

$$\boxed{T(t) = \frac{r'(t)}{\|r'(t)\|}}$$

>> Unit normal vector [or Principal unit normal vector].

$$\boxed{N(t) = \frac{T'(t)}{\|T'(t)\|}}$$

> Find the Principal unit normal vector for the circular helix $x = a\cos t$, $y = a\sin t$, $z = ct$?

$\Rightarrow x = a\cos t$, $y = a\sin t$, $z = ct$

$r(t) = a\cos t\, \hat{i} + a\sin t\, \hat{j} + ct\, k$

$r'(t) = -a\sin t\, \hat{i} + a\cos t\, \hat{j} + c\, k$

$\|r'(t)\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + c^2} = \sqrt{a^2\sin^2 t + a^2\cos^2 t + c^2}$

$= \sqrt{a^2[\sin^2 t + \cos^2 t] + c^2} = \sqrt{a^2 \times 1 + c^2} = \sqrt{a^2 + c^2}$

$$\therefore T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{-a\sin t\, i + a\cos t\, j + cK}{\sqrt{a^2+c^2}}$$

$$T(t) = \frac{-a\sin t}{\sqrt{a^2+c^2}} i + \frac{a\cos t}{\sqrt{a^2+c^2}} j + \frac{c}{\sqrt{a^2+c^2}} K$$

$$T'(t) = \frac{-a\cos t}{\sqrt{a^2+c^2}} i - \frac{a\sin t}{\sqrt{a^2+c^2}} j.$$

$$\|T'(t)\| = \sqrt{\left[\frac{-a\cos t}{\sqrt{a^2+c^2}}\right]^2 + \left[\frac{-a\sin t}{\sqrt{a^2+c^2}}\right]^2}$$

$$= \sqrt{\frac{a^2\cos^2 t}{a^2+c^2} + \frac{a^2\sin^2 t}{a^2+c^2}}$$

$$= \sqrt{\frac{a^2\cos^2 t + a^2\sin^2 t}{a^2+c^2}} = \sqrt{\frac{a^2[\cos^2 t + \sin^2 t]}{a^2+c^2}}$$

$$= \sqrt{\frac{a^2}{a^2+c^2}} = \frac{\sqrt{a^2}}{\sqrt{a^2+c^2}} = \underline{\underline{\frac{a}{\sqrt{a^2+c^2}}}}.$$

$$\therefore N(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{\frac{-a\cos t}{\sqrt{a^2+c^2}} i - \frac{a\sin t}{\sqrt{a^2+c^2}} j}{\frac{a}{\sqrt{a^2+c^2}}}$$

$$= \frac{\frac{-a\cos t}{\sqrt{a^2+c^2}}}{\frac{a}{\sqrt{a^2+c^2}}} i - \frac{\frac{a\sin t}{\sqrt{a^2+c^2}}}{\frac{a}{\sqrt{a^2+c^2}}} j$$

$= -\cos t\, \hat{i} - \sin t\, \hat{j}$

) Find the unit tangent vector and unit normal vector for the curve defined by $r(t)$ at the given point:

1) $r(t) = e^t \cos t\, \hat{i} + e^t \sin t\, \hat{j} + 2K$; $t = 0$

2) $r(t) = (t^2+1)\, \hat{i} + t\, \hat{j}$; $t = 1$

3) $r(t) = \frac{1}{2} t^2\, \hat{i} + \frac{1}{3} t^3\, \hat{j}$; $t = 2$

4) $r(t) = 5\cos t\, \hat{i} + 5\sin t\, \hat{j}$; $t = \pi/3$

5) $r(t) = 4\cos t\, \hat{i} + 4\sin t\, \hat{j} + tK$; $t = \pi/2$