Improper Integrals.

The integral $\int_a^b f(x)\,dx$ is called improper integral if.

(i) $a = -\infty$ or $b = \infty$ or both.

Definitions

Suppose that for a fixed real number a, f is integrable on $[a,b]$ $\forall\ b > a$. Then, we define

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx.$$

If the limit on the right hand side is finite we say that the improper integral converges and the limit is the value of the integral. If the limit fails to exist the improper integral is said to diverge.

Similarly we define

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_{a}^{b} f(x)\,dx.$$

Finally if f is integrable on $[a,b]$ $\forall\, a<b$, then we define

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx.$$

and the improper integral on the left hand side is convergent if and only if both improper integrals on the right hand side exists in accordance with the definitions given above and otherwise is divergent.

eg 1: Evaluate $\int_{1}^{\infty} 1/x^2\,dx$

$$= \lim_{b \to \infty} \int_{1}^{b} 1/x^2\,dx \quad \text{(By defn)}$$

$$= \lim_{b \to \infty} \int_1^b (x^{-2})\, dx$$

$$= \lim_{b \to \infty} \left[\frac{x^{-1}}{-1} \right]_1^b$$

$$= \lim_{b \to \infty} \left[-1/x \right]_1^b$$

$$= \lim_{b \to \infty} \left[-1/b - -1 \right]$$

$$= \lim_{b \to \infty} 1 - 1/b$$

$$= \underline{\underline{1}} \quad \left(\text{since } \lim_{b \to \infty} 1/b = 0 \right)$$

---

2 $$\int_{-\infty}^{0} \frac{dx}{(1-3x)^2} = \lim_{a \to -\infty} \int_a^0 \frac{dx}{(1-3x)^2}$$

$$= \lim_{a \to -\infty} \int_a^0 (1-3x)^{-2}\, dx$$

$$= \lim_{a \to -\infty} \left[\frac{(1-3x)^{-1}}{-1 \times -3} \right]_a^0$$

$$= \lim_{a \to -\infty} \frac{1}{3}\left[\frac{1}{1-3x}\right]_a^0$$

$$= \lim_{a \to -\infty} \frac{1}{3}\left[\frac{1}{1-0} - \frac{1}{1-3a}\right]$$

$$= \frac{1}{3}\left(1 - \lim_{a \to -\infty} \frac{1/a}{1/a - 3}\right)$$

$$= \frac{1}{3}\left(1 - \frac{0}{0-3}\right)$$

$$= \underline{\underline{1/3}}$$

Since the limit is finite, the given improper integral is convergent and has the value $\frac{1}{3}$.

3 Evaluate $\int_{-\infty}^{0} \cosh x \, dx$.

$$\int_{-\infty}^{0} \cosh x \, dx = \lim_{a \to -\infty} \int_{a}^{0} \cosh x \, dx$$

$$= \lim_{a \to -\infty} [\sinh x]_a^0$$

$$= \lim_{a \to -\infty} [\sinh 0 - \sinh a]$$

$$= \lim_{a \to -\infty} 0 - \left(\frac{e^a - e^{-a}}{2}\right]$$

[we know that $\lim_{x \to -\infty} e^x = 0$ and

$\lim_{x \to \infty} e^x = \infty$] $\left(\lim_{x \to \infty} 1/e^{-x} = \infty\right)$

$$= 0 - (\cancel{\infty} \; 0$$

$$= 0 - \left(0 - \frac{\infty}{2}\right)$$

$$= \infty$$

Since the limit is infinite, the given improper integral is divergent.

~~4) Evaluate $\int_{-\infty}^{0} \cosh x \, dx$~~

~~$= \lim_{a \to -\infty} \int_{a}^{0} \cos$~~

4) Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ is convergent.

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}.$$

$$= \lim_{a \to -\infty} \int_{a}^{0} \frac{dx}{1+x^2} + \lim_{b \to \infty} \int_{0}^{b} \frac{dx}{1+x^2}.$$

$$= \lim_{a \to -\infty} \left[\tan^{-1} x\right]_{a}^{0} + \lim_{b \to \infty} \left[\tan^{-1} x\right]_{0}^{b}.$$

$$= \lim_{a \to \infty} \left[\tan^{-1} 0 - \tan^{-1} a\right] + \lim_{b \to \infty} \left[\tan^{-1} b - \tan^{-1} 0\right]$$

$$= \lim_{a \to -\infty} (0 - \tan^{-1} a) + \lim_{b \to \infty} \left[\tan^{-1} b - 0\right].$$

$$= 0 - (-\pi/2) + (\pi/2 - 0)$$

$$= 2\pi/2 = \pi \,//$$

Since both limits are finite, the given improper integral is convergent and has the value π.

5) Evaluate $\int_1^{\infty} \frac{\ln x}{x^2} dx$.

$$\int_1^{\infty} \frac{\ln x}{x^2} dx = \lim_{b \to \infty} \int_1^{b} \frac{\ln x}{x^2} dx.$$

$$= \lim_{b \to \infty} \left(\ln x \times -1/x - \int 1/x \cdot x^{-1}/x \, dx \right]_1^b$$

(product rule for integrals)

$$= \lim_{b \to \infty} \left[\ln x \times -1/x + -1/x \right]_1^b$$

$$= \lim_{b \to \infty} \left(-\frac{\ln b}{b} - 1/b - (0-1) \right)$$

$$= \lim_{b \to \infty} -\frac{\ln b}{b} - 0 + 1$$

$$= 1 - \lim_{b \to \infty} \frac{\ln b}{b}$$ (∞/∞ form, Then apply l'Hopital's rule)

$$= 1 - \lim_{b \to \infty} \frac{1/b}{1} = 1 - 0 = 1$$

6. S.T $\int_{-\infty}^{\infty} \frac{2x}{1+x^2} dx$ is divergent.

By defn.

$$\int_{-\infty}^{\infty} \frac{2x}{1+x^2} dx = \int_{-\infty}^{0} \frac{2x}{1+x^2} dx + \int_{0}^{\infty} \frac{2x}{1+x^2} dx.$$

[By substitute put $u = 1+x^2$.
$du = 2x\,dx$

$$\therefore \int_{-\infty}^{0} \frac{2x}{1+x^2} dx = \int_{-\infty}^{0} \frac{du}{u} = [\ln u]_{-\infty}^{0}$$
$$= [\ln(1+x^2)]_{-\infty}^{0}$$]

$$= \lim_{a \to -\infty} [\ln(1+x^2)]_a^0 + \lim_{b \to \infty} [\ln(1+x^2)]_0^b$$

$$= \lim_{a \to -\infty} [\ln(1+0) - \ln(1+a^2)] + \lim_{b \to \infty} [\ln(1+b^2) - \ln 1]$$

$$= \lim_{a \to -\infty} (0 - \ln(1+a^2)) + \lim_{b \to \infty} (\ln(1+b^2))$$

It doesnot exist.

$\therefore$ The given integral diverges.

7 S.T $\int_a^\infty 1/x^p \, dx$ where p is a constant and $a>0$, converges if $P>1$ and diverges if $P \le 1$.

$$\int_a^\infty 1/x^p = \lim_{b\to\infty} \int_a^b 1/x^p \, dx \quad \text{(By defn)}$$

$$= \lim_{b\to\infty} \left[\frac{x^{-P+1}}{-P+1}\right]_a^b \quad \text{Provided } P \ne 1$$

$$= \lim_{b\to\infty} \left(\frac{b^{-P+1}}{-P+1} - \frac{a^{-P+1}}{-P+1}\right)$$

$$= \frac{1}{1-P} \lim_{b\to\infty} \left(b^{1-P} - a^{1-P}\right)$$

$=$

$\lim_{b\to\infty} b^{1-P} \to 0$ if $P>1$.

$\to \infty$ if $P<1$.

$\therefore \int_a^\infty 1/x^p$ converges if $p > 1$

diverges if $p \leq 1$.

If $p = 1$ $\int_a^\infty 1/x^p \, dx = \int_a^\infty 1/x \, dx$

$= \lim_{b \to \infty} \int_a^b 1/x \, dx$

$= \lim_{b \to \infty} [\ln x]_a^b$

$= \lim_{b \to \infty} (\ln b - \ln a)$

$\longrightarrow \infty$

$\therefore$ It diverges.

8. S.T $\int_0^\infty e^{-tx} \, dx$ where t is a constant.

converges if $t > 0$ and diverges if ~~t~~ $t \leq 0$.

Soln:- $\int_0^\infty e^{-tx} \, dx = \lim_{b \to \infty} \int_0^b e^{-tx} \, dx$.

$$= \lim_{b\to\infty}\left[\frac{e^{-tx}}{-t}\right]_0^b.$$

$$= \lim_{b\to\infty}\left(\frac{e^{-tb} - e^{0}}{-t}\right)$$

$$= \lim_{b\to\infty} \frac{1}{t}\left(1 - e^{-tb}\right)$$

$$= \frac{1}{t}\left[1 - \lim_{b\to\infty} e^{-tb}\right]$$

But $\lim_{b\to\infty} e^{-tb} = \begin{cases} 0 & \text{if } t>0 \\ \infty & \text{if } t<0 \end{cases}$

$\therefore \int_0^\infty e^{-tx}\,dx$ convgs if $t>0$ and

dvgs if $t<0$

$\lim_{x\to\infty} e^{x} = \infty$

$\lim_{x\to-\infty} e^{x} = 0.$