INDETERMINATE FORMS

In limits and continuity it was shown that: If $L = \lim_{x \to c} f(x)$ and $M = \lim_{x \to c} g(x)$ and if $M \neq 0$ then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = L/M.$$

However if $M = 0$ and $L \neq 0$, we usualy take the limit to be infinite (when it exist).

In the case $L = 0$, $M = 0$ the limit of the quotient f/g is said to be 'indeterminate'.

Thm :- L'Hopital's rule - Stronger form

Suppose that $f(a) = g(a) = 0$ and that f and g are differentiable on an open interval I containing a. Suppose also that $g'(x) \neq 0$ on I

If $x \neq a$ Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

If the limit exist on the right exist.

Problem :

Evaluate $\lim_{x \to 1} \frac{1-x}{\ln x}$.

$\lim_{x \to 1} \frac{1-x}{\ln x}$ ($0/0$ form apply l'Hopital's rule)

$= \lim_{x \to 1} \frac{0-1}{1/x}$

$= \lim_{x \to 1} -x = -1$

(2) $\lim_{x \to 0} \frac{x - \sin x}{x^3}$ ($0/0$ form) Apply l'Hopitals rule

$= \lim_{x \to 0} \frac{1-\cos x}{3x^2}$ (again $0/0$ form)

$$= \lim_{x \to 0} \frac{0 - \sin x}{6x}$$

$$= \lim_{x \to 0} \frac{\sin x}{6x}$$ (0/0 form Apply l'Hopitals rule,

$$= \lim_{x \to 0} \frac{\cos x}{6}$$

$$= 1/6$$

3. Evaluate $\lim_{x \to 0} \frac{1 - \cos x}{x + x^2}$.

4 Evaluate $\lim_{x \to 0} \frac{\sin x}{x^2}$.

5 Evaluate $\lim_{x \to 0} \frac{x^9}{1 - \cos x}$.

The indeterminate form ∞/∞

eg:- Evaluate the following limit.

i) $\lim_{x \to \pi/2} \left(\frac{\sec x}{1+\tan x} \right)$ (∞/∞ form) $\left\{ \begin{array}{l} \text{as } x \to \pi/2 \\ \quad \sec x \to \infty \\ \& \\ x \to \pi/2 \quad 1+\tan \pi/2 \\ \quad \to \infty. \end{array} \right.$

$= \lim_{x \to \pi/2} \frac{\sec x \tan x}{0+\sec^2 x}$ (By l'Hopitals rule)

$= \lim_{x \to \pi/2} \frac{\tan x}{\sec x}$ $\qquad \tan x = \frac{\sin x}{\cos x}$

$= \lim_{x \to \pi/2} \sin x = \sin \pi/2 = \underline{\underline{1}}$

ii) $\lim_{x \to 0^+} \frac{\ln x}{\cot x}$ $\left(\begin{array}{l} x \to 0 \quad \ln x \to \infty. \\ \cot 0 \to \infty, \end{array} \right.$

∞/∞ form.

$= \lim_{x \to 0} \frac{1/x}{-\csc^2 x}$

$= \lim_{x \to 0} \frac{1}{1/x} \times -\sin^2 x$ $\qquad \left(\because \frac{1}{\csc} = \sin \right)$

$$= \lim_{x \to 0} \frac{-\sin^2 x}{x} \quad (0/0 \text{ form})$$

(Again By l'Hopital rule)

$$= \lim_{x \to 0} \frac{-2\sin x \cos x}{1}$$

$$= \underline{\underline{0}}$$

iii) $\lim_{x \to \infty} \frac{\ln x}{2\sqrt{x}}$

Indeterminate Difference

$\infty - \infty$ form

i $\lim_{x \to 0} \left(\frac{1}{\sin x} - 1/x \right)$ ($\infty - \infty$ form)

$$= \lim_{x \to 0} \frac{x - \sin x}{x \sin x} \quad (0/0 \text{ form})$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{x \cos x + \sin x} \quad (0/0 \text{ form})$$

(By l'Hopital rule)

$$= \lim_{x \to 0} \frac{\sin x}{\cos x + \cos x + x \times -\sin x}$$

$$= \lim_{x \to 0} \frac{\sin x}{2\cos x - x\sin x}$$

$$= \frac{0}{2-0} = \underline{\underline{0}}.$$

3) $\lim_{x \to 0} \frac{1-\cos x}{x + x^2}$. ($0/0$ form)

$$= \lim_{x \to 0} \frac{0 - -\sin x}{1 + 2x}$$

$$= \frac{0}{1 + 2 \times 0} = 0/1$$

$$= 0 //.$$

(4) $\lim_{x \to 0} \frac{x^2}{1-\cos x}$ $0/0$ form

$$= \lim_{x \to 0} \frac{2x}{0-\sin x} \quad (0/0 \text{ form})$$

$$= \lim_{x \to 0} \frac{2}{\cos x}$$

$$= 2/1 = 2$$

I) Evaluate the following limits

i) $\lim_{x \to 2} \frac{x-2}{x^2-4}$.

ii) $\lim_{x \to 0} \frac{8x^2}{\cos x - 1}$.

iii) $\lim_{x \to \pi/2} x \tan x - \frac{1}{2}\pi \sec x$.

Indeterminate Products: $(0 \cdot \infty)$.

eg:- Evaluate $\lim_{x \to 0} x \cot x$.

$$\lim_{x \to 0} x \cot x \quad (0 \cdot \infty \text{ form}).$$

[Now we rewrite $x \cot x$ into ∞/∞ or $0/0$ form].

$$= \lim_{x \to 0} x \cdot \frac{1}{\tan x} \quad (0/0 \text{ form})$$

$$= \lim_{x \to 0} \frac{1}{\sec^2 x} = 1/1 = 1 \,//.$$

(2) $\lim_{x \to 0} (x \ln x)$ $\quad (0 \cdot \infty$ form)

$$= \lim_{x \to 0^+} \frac{\ln x}{1/x} \quad \left(\frac{1}{1/x} = x\right)$$

$(\infty/\infty$ form)

$$= \lim_{x \to 0} \frac{1/x}{-1/x^2}$$

$$= \lim_{x \to 0} -\frac{1}{x} \times \frac{x^2}{1}$$

$$= \lim_{x \to 0} -x = \underline{\underline{0}}.$$

(3) Indeterminate Powers (1^∞, 0^0 & ∞^0)

Evaluate $\lim_{x \to \infty} x^{1/x}$. $\left[\lim_{x \to \infty} 1/x = 0 \right.$

(∞^0 form)

Let $k(x) = x^{1/x}$. Then

$$\ln k(x) = \ln x^{1/x}$$

$$= \frac{1}{x} \ln x \quad (\because \ln x^n = n \ln x)$$

$$\lim_{x \to \infty} \ln k(x) = \lim_{x \to \infty} \frac{\ln x}{x} \quad (\infty/\infty \text{ form})$$

$$= \lim_{x \to \infty} \frac{1/x}{1}$$

$$= \lim_{x \to \infty} 1/x = 0$$

$$\therefore \lim_{x \to \infty} e^{\ln k(x)} = e^0$$

$$\lim_{x \to \infty} k(x) = 1 \Rightarrow \lim_{x \to \infty} x^{1/x} = 1$$

4) Evaluate $\lim_{x\to 0} (\cos x)^{1/x^2}$.

Let $k(x) = (\cos x)^{1/x^2}$.

$\ln k(x) = \ln (\cos x)^{1/x^2}$

$= \frac{1}{x^2} \ln \cos x$.

$\lim_{x\to 0} \ln k(x) = \lim_{x\to 0} \frac{\ln \cos x}{x^2}$

$(0/0 \text{ form})$

$= \lim_{x\to 0} \frac{\frac{1}{\cos x} \cdot -\sin x}{2x}$.

$= \lim_{x\to 0} \frac{-\tan x}{2x}$ $(0/0$ form

Apply l' Hospital rules

$= \lim_{x\to 0} \frac{-\sec^2 x}{2}$.

$= -1/2$

$\lim_{x\to 0} e^{\ln k(x)} = e^{-1/2}$.

$\Rightarrow \lim_{x\to 0} k(x) = e^{-1/2} \Rightarrow \lim_{x\to 0} (\cos x)^{1/x^2} = e^{-1/2} = 1/\sqrt{e}$