Chapter - 2

Multiple Integrals

Multiple integrals

The process of integration for one variable can be extended to the functions of more than one variable. This generalization of definite integrals is known as 'multiple integrals'.

Fubini's Theorem [First Form]

If $f(x,y)$ is continuous throughout the rectangular region $R: a \le x \le b,\ c \le y \le d$ then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

1) If $f(x,y) = x^2y - 2xy$ and $R: 0 \le x \le 3,\ -2 \le y \le 0$, then evaluate $\iint_R f(x,y)\,dA$?

$f(x,y) = x^2y - 2xy$

$$\iint_R f(x,y)\,dA = \int_0^3 \int_{-2}^{0} (x^2y - 2xy)\,dy\,dx$$

$$= \int_0^3 \left[\frac{x^2 y^2}{2} - \frac{2x y^2}{2} \right]_{-2}^{0} dx$$

$$= \int_0^3 \left[\frac{x^2 y^2}{2} - x y^2 \right]_{-2}^{0} dx$$

$$= \int_0^3 \left[\left(\frac{x^2 \times 0^2}{2} - x \times 0^2 \right) - \left(\frac{x^2 \times (-2)^2}{2} - x \times (-2)^2 \right) \right] dx$$

$$= \int_0^3 0 - \left(\frac{x^2 \times 4}{2} - 4x \right) dx$$

$$= \int_0^3 (-2x^2 + 4x)\, dx$$

$$= \left. -\frac{2 \times x^3}{3} + \frac{4 \times x^2}{2} \right]_0^3 = \left. -\frac{2x^3}{3} + 2x^2 \right]_0^3$$

$$= \left(\frac{-2 \times (3)^3}{3} + 2 \times (3)^2 \right) - \left(\frac{-2 \times 0^3}{3} + 2 \times 0^2 \right)$$

$$= \left(\frac{-2 \times 27}{3} + 2 \times 9 \right) - 0 = -2 \times 9 + 2 \times 9 = 0 //$$

1) Evaluate $\int_0^1 \int_0^2 xy(x-y)\,dx\,dy$

2) Evaluate $\int_0^3 \int_0^2 (4-y^2)\,dy\,dx$

3) $\int_{-1}^0 \int_{-1}^1 (x+y+1)\,dx\,dy$

4) $\int_0^1 \int_1^2 (x^2+y^2)\,dx\,dy$

5) $\int_1^{\ln 8} \int_0^{\ln y} e^{x+y}\,dx\,dy$.

Fubini's Theorem [Stronger Form]

Let $f(x,y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $r(x) \le y \le s(x)$ with r and s are continuous on $[a,b]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{r(x)}^{s(x)} f(x,y)\,dy\,dx$$

2. If R is defined by $c \le y \le d$, $g(y) \le x \le h(y)$, with g and h are continuous on $[a,b]$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy.$$

Properties of Double Integrals

If $f(x,y)$ and $g(x,y)$ are continuous, then

1) Constant Multiple $\Rightarrow$

$$\iint_R c\,f(x,y)\,dA = c\iint_R f(x,y)\,dA$$

2) Sum and Difference $\Rightarrow$

$$\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$$

$$\iint_R [f(x,y)-g(x,y)]\,dA = \iint_R f(x,y)\,dA - \iint_R g(x,y)\,dA$$

3) Domination $\Rightarrow$

(a) $\iint_R f(x,y)\,dA \geq 0$ if $f(x,y) \geq 0$ on R

(b) $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$ if $f(x,y) \geq g(x,y)$ on R.

4) Additivity $\Rightarrow$

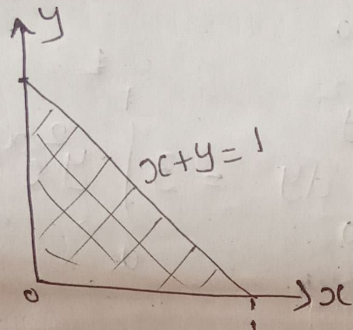
$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA.$$

Sketch the region of integration and evaluate :

(a) $\int_0^1 \int_0^{1-x} (x^2+y^2)\, dy\, dx$ (b) $\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\, dx\, dy$.

$\Longrightarrow$

(a) The region of integration is $0 \leq x \leq 1$, $0 \leq y \leq 1-x$.



$$\int_0^1 \int_0^{1-x} (x^2+y^2)\, dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^{1-x} dx$$

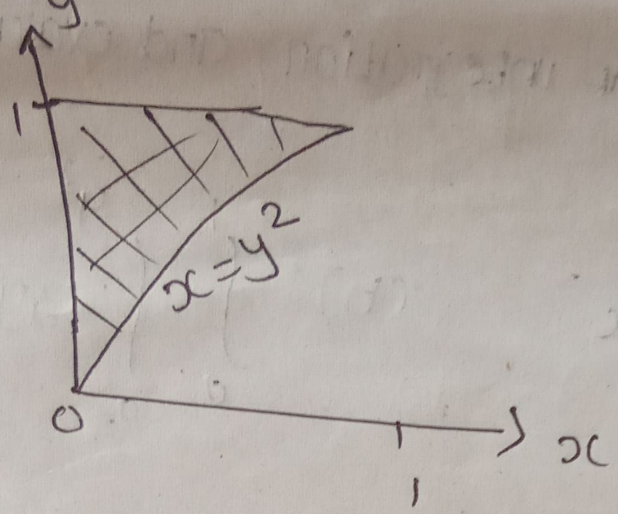
$$= \int_0^1 \left[\left(x^2(1-x) + \frac{(1-x)^3}{3} \right) - 0 \right] dx$$

$$= \int_0^1 \left[x^2 - x^3 + \frac{(1-x)^3}{3} \right] dx$$

$$= \left. \frac{x^3}{3} - \frac{x^4}{4} + \frac{(1-x)^4}{\frac{3 \times 4}{-1}} \right]_0^1 = \left. \frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12} \right]_0^1$$

$$= \left(\frac{1}{3} - \frac{1}{4} \right) - \left(0 - 0 - \frac{1}{12} \right) = \frac{1}{3} + \frac{1}{12} - \frac{1}{4} = \frac{2}{12} = 1/6$$

(b)



The region of integration is $0 \le y \le 1$, $0 \le x \le y^2$

$$\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\,dx\,dy = 3\int_0^1 \int_0^{y^2} y^3 e^{xy}\,dx\,dy$$

$$= 3\int_0^1 \left[\frac{y^3 e^{xy}}{y}\right]_0^{y^2} dy = 3\int_0^1 \left[y^2 e^{xy}\right]_0^{y^2} dy$$

$$= 3\int_0^1 \left[\left(y^2 e^{y^2 \cdot y}\right) - y^2\right] dy$$

$$= 3\int_0^1 (y^2 e^{y^3} - y^2)\,dy$$

$$= \int_0^1 (3y^2 e^{y^3} - 3y^2)\,dy$$

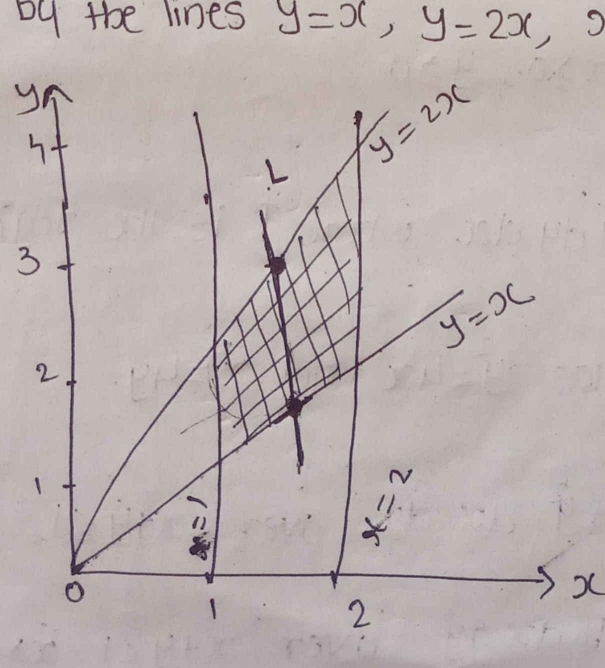
$$= \int_0^1 3y^2 e^{y^3}\,dy - \int_0^1 3y^2\,dy$$

$$= e^{y^3}\Big]_0^1 - \frac{3\times y^3}{3}\Big]_0^1 \qquad \left[\because \frac{d}{dx}e^{y^3} = e^{y^3}\times 3y^2 \quad \therefore \int e^{y^3}\times 3y^2 = e^{y^3}\right]$$

$$= (e^{1^3} - e^{0^3}) - (1^3 - 0^3)$$

$$= (e^1 - 1) - 1 = \underline{e-2}$$

› Integrate $f(x,y) = \frac{x}{y}$ over the region in the first quadrant bounded by the lines $y = x$, $y = 2x$, $x = 1$, $x = 2$?



$$\iint_R F(x,y)\,dA = \int_1^2 \int_x^{2x} \frac{x}{y}\,dy\,dx$$

$$= \int_1^2 \left[x \ln y \right]_x^{2x} dx = \int_1^2 x(\ln 2x - \ln x)\,dx$$

$$= \int_1^2 x\left[\ln\left(\frac{2x}{x}\right)\right] dx \qquad \left(\because \log\frac{a}{b} = \log a - \log b\right)$$

$$= \int_1^2 x \ln 2\,dx = \ln 2 \int_1^2 x\,dx$$

$$= \ln 2 \times \left.\frac{x^2}{2}\right]_1^2 = \ln 2\left(\frac{4}{2} - \frac{1}{2}\right) = \frac{3}{2}\ln 2$$

1) Evaluate $\iint_R xy\,dx\,dy$ where R is the region $x^2+y^2 \le a^2$, $x \ge 0$, $y \ge 0$

2) Evaluate $\iint_R y\,dy\,dx$ where R is the region bounded by the Parabolas $y^2 = 4x$ and $x^2 = 4y$.

3) Evaluate $\iint x^2+y^2\,dx\,dy$ over $x+y \le 1$, $x \ge 0$, $y \ge 0$.

4) Evaluate $\iint xy\,dx\,dy$ over $x+y \le 1$, $x \ge 0$, $y \ge 0$.