Extreme Values and Saddle Points

Local Extrema

Let $f(x,y)$ be defined on a region R containing the point (a,b). Then

(i) $f(a,b)$ is a 'local maximum' value of f if $f(a,b) \geq f(x,y)$

(ii) $f(a,b)$ is a 'local minimum' value of f if $f(a,b) \leq f(x,y)$

First derivative test for local extreme values

If $f(x,y)$ has a local maximum or minimum value at an interior point (a,b) of its domain and if the first partial derivatives exist there, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

Critical Point

An interior point of the domain of a function $f(x,y)$ where both f_x and f_y are zero or one or both of f_x and f_y do not exist is a 'critical point' of f.

Saddle Point

A differentiable function $f(x,y)$ has a 'saddle point' at a critical point (a,b) if there are domain points (x,y) where $f(x,y) > f(a,b)$ and domain points (x,y) where $f(x,y) < f(a,b)$

Second derivative Test

Suppose that $f(x,y)$ and its first and second Partial derivatives are continuous throughout a disk centered at (a,b) and that $f_x(a,b) = f_y(a,b) = 0$. Then

(i) f has local maximum at (a,b) if

$$f_{xx} < 0 \text{ and } f_{xx}f_{yy} - f_{xy}^2 > 0 \text{ at } (a,b)$$

(ii) f has a local minimum at (a,b) if

$$f_{xx} > 0 \text{ and } f_{xx}f_{yy} - f_{xy}^2 > 0 \text{ at } (a,b)$$

(iii) f has a saddle Point at (a,b) if

$$f_{xx}f_{yy} - f_{xy}^2 < 0 \text{ at } (a,b)$$

(iv) Test is inconclusive at (a,b) if

$$f_{xx}f_{yy} - f_{xy}^2 = 0 \text{ at } (a,b).$$

Remark

The expression $f_{xx}f_{yy} - f_{xy}^2$ is called the 'discriminant' of f.

Find all local maxima, local minima and saddle points of the following functions:

(i) $f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$

$\Rightarrow f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$

$f_x = 2x + y + 3$

$f_{xx} = 2$

$f_{xx} = 2 > 0.$

$f_y = x + 2y - 3$

$f_{yy} = 2$

$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x + 2y - 3) = 1$

$\therefore f_{xx} f_{yy} - f_{xy}^2 = 2 \times 2 - (1)^2 = 4 - 1 = \underline{\underline{3}} > 0$

$\because f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$, f has local minimum.

$f_x = 0$ and $f_y = 0 \Rightarrow 2x + y + 3 = 0$, $x + 2y - 3 = 0$

$\Rightarrow 2x + y + 3 = 0 \quad - (1)$

$x + 2y - 3 = 0 \quad - (2)$

$2 \times (2) \Rightarrow 2x + 4y - 6 = 0 \quad - (3)$

$(1) - (3) \Rightarrow$

$$\begin{array}{r} 2x + y + 3 = 0 \\ 2x + 4y - 6 = 0 \\ \hline -3y + 9 = 0 \end{array}$$

$-3y = -9$

$y = \frac{-9}{-3} = \underline{\underline{3}}$

$\therefore (1) \Rightarrow 2x + 3 + 3 = 0 \Rightarrow 2x + 6 = 0 \Rightarrow 2x = -6 \Rightarrow x = \frac{-6}{2} = \underline{\underline{-3}}$

$\therefore f$ attain local minima value at $(-3,3)$ and the minimum value is $f(-3,3) = (-3)^2 + 3\times -3 + (3)^2 + 3\times -3 - 3\times 3 + 4$

$= 9 - 9 + 9 - 9 - 9 + 4$

$= -5$

(ii) $f(x,y) = x^3 + y^3 - 3x - 12y + 20$.

$\Rightarrow f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$f_x = 3x^2 - 3$

$f_{xx} = 6x$

$f_y = 3y^2 - 12$

$f_{yy} = 6y$

$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3y^2 - 12) = 0$

$f_{xx} f_{yy} - f_{xy}^2 = 6x \times 6y - 0 = 36xy$

$f_x = 0$ and $f_y = 0 \Rightarrow 3x^2 - 3 = 0$, $3y^2 - 12 = 0$

$\Rightarrow 3x^2 = 3$, $3y^2 = 12$

$\Rightarrow x^2 = 1$, $y^2 = \frac{12}{3} = 4$

$\Rightarrow x = \pm 1$, $y^2 = \pm 2$

$\therefore$ The critical Points are $(1,2), (-1,2), (1,-2), (-1,-2)$.

At $(1,2) \Rightarrow f_{xx} = 6x = 6 \times 1 = 6 > 0$

$f_{xx} f_{yy} - f_{xy}^2 = 36xy = 36 \times 1 \times 2 = 72 > 0$

$\because f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$, f has local minima at $(1,2)$ and the value is $f(1,2) = (1)^3 + (2)^3 - 3 \times 1 - 12 \times 2 + 20$

$= 1 + 8 - 3 - 24 + 20 = \underline{\underline{2}}$

At $(-1,2) \Rightarrow f_{xx} = 6x = 6 \times -1 = -6 < 0$

$f_{xx} f_{yy} - f_{xy}^2 = 36xy = 36 \times -1 \times 2 = -72 < 0$

$\therefore f$ has a saddle Point.

At $(1,-2) \Rightarrow f_{xx} = 6x = 6 \times 1 = 6 > 0$

$f_{xx} f_{yy} - f_{xy}^2 = 36xy = 36 \times 1 \times -2 = -72 < 0$

$\therefore f$ has a saddle Point.

At $(-1,-2) \Rightarrow f_{xx} = 6x = 6 \times -1 = -6 < 0$

$f_{xx} f_{yy} - f_{xy}^2 = 36xy = 36xy = 36 \times -1 \times -2 = 72 > 0$.

$\because f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$, f has local maxima at $(-1,-2)$ and the value is $f(-1,-2) = (-1)^3 + (-2)^3 - 3 \times -1 - 12 \times -2 + 20$

$= -1 - 8 + 3 + 24 + 20 = 38$ //

› Find all local maxima, local minima and saddle points of the following functions :

1) $f(x,y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$

2) $f(x,y) = x^2 + 3xy + 3y^2 - 6x + 3y - 6$

3) $f(x,y) = x^2 + xy + 3x + 2y + 5$

4) $f(x,y) = 5xy - 7x^2 + 3x - 6y + 2$

5) $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$.