Lagrange Multipliers

Method of lagrange multipliers

The method says that the extreme values of the function $f(x,y,z)$ whose variables are subject to a constraint $g(x,y,z)=0$ are to be found on the surface $g=0$ at a point where $\nabla f = \lambda \nabla g$ for some scalar λ.

[λ called Lagrange multiplier]

Theorem [The orthogonal Gradient Theorem]

Suppose that $f(x,y,z)$ is differentiable in a region whose interior contains a smooth curve

$C: r(t) = g(t)i + h(t)j + k(t)k$.

If P_0 is a point on C where f has a local maximum or minimum relative to its values on C, then ∇f is orthogonal to C at P_0.

Proof

we have $r(t) = g(t)i + h(t)j + k(t)k$.

on comparing this equation with $r(t) = x(t)i + y(t)j + z(t)k$, we get $x(t) = g(t)$, $y(t) = h(t)j$, $z(t) = k(t)$

$\therefore f(x,y,z) = f(g(t), h(t), k(t))$.

By chain rule we get,

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dg}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dh}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dk}{dt} \quad \text{— (1)}$$

The gradient vector of $f(x,y,z)$ at (x,y,z) is

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k \quad \text{— (2)}$$

The velocity vector is given by

$$V = \frac{dr}{dt}$$

$$V = \frac{dg}{dt} i + \frac{dh}{dt} j + \frac{dk}{dt} k \quad \text{— (3)}$$

$$\nabla f \cdot V = \left[\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k\right] \cdot \left[\frac{dg}{dt} i + \frac{dh}{dt} j + \frac{dk}{dt} k\right]$$

$$\nabla f \cdot V = \frac{\partial f}{\partial x} \cdot \frac{dg}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dh}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dk}{dt}$$

$$\nabla f \cdot V = \frac{df}{dt} \quad \text{— (4)}$$

Suppose that at P_0 on C, f has local max or min. relative to it's value on C. The $\frac{df}{dt} = 0$

$\therefore$ (4) $\Rightarrow$ $\nabla f \cdot V = 0$. $\therefore$ ∇f is orthogonal to C.

Lagrange multiplier

Method of Lagrange multipliers

Suppose that $f(x,y,z)$ and $g(x,y,z)$ are differentiable. To find the local max. and min. values of f subject to the constraint $g(x,y,z)=0$, find the values of x,y,z and λ that simultaneously satisfy the equations $\nabla f = \lambda \nabla g$ and $g(x,y,z)=0$. [where λ called Lagrange multiplier]

Find the greatest and smallest values that the function $f(x,y)=xy$ takes on the ellipse $x^2+2y^2=1$?

$$f(x,y) = xy$$

$$g(x,y) = 0 \Rightarrow x^2+2y^2-1=0$$

~~First we have to find~~.

We first find values of x, y and λ for which

$$\nabla f = \lambda \nabla g \text{ and } g(x,y)=0.$$

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$$

$$\nabla f = y i + x j$$

$$\nabla g = \frac{\partial g}{\partial x} i + \frac{\partial g}{\partial y} j$$

$$\nabla g = 2x i + 4y j$$

$$\therefore \nabla f = \lambda \nabla g \Rightarrow y i + x j = \lambda [2x i + 4y j]$$

$$\Rightarrow y i + x j = 2x\lambda i + 4y\lambda j$$

on comparing both sides we get

$$y = 2x\lambda \text{ — (1) and } \quad x = 4y\lambda \text{ — (2)}$$

$$\therefore (1) \Rightarrow y = 2 \times (4y\lambda)\lambda$$

$$\Rightarrow y = 8y\lambda^2$$

$$\Rightarrow y - 8y\lambda^2 = 0 \Rightarrow y[1 - 8\lambda^2] = 0$$

$ab = 0 \Rightarrow a = 0$ or $b = 0$

$$\Rightarrow y = 0 \text{ or } 1 - 8\lambda^2 = 0$$

$$\Rightarrow y = 0 \text{ or } 1 = 8\lambda^2$$

$$\Rightarrow y = 0 \text{ or } \lambda^2 = 1/8$$

$$\Rightarrow y = 0 \text{ or } \lambda = \pm \frac{1}{2\sqrt{2}}$$

If $y = 0$, (2) $\Rightarrow x = 4 \times 0 \times \lambda \Rightarrow x = 0$.

$\because g(x,y) \neq 0$, $(0,0)$ is not a Point of the given ellipse. Hence $y \neq 0$.

If $\lambda = \pm \frac{1}{2\sqrt{2}}$, ② $\Rightarrow x = 4\lambda y = 4 \times \pm \frac{1}{2\sqrt{2}} y$

$\Rightarrow x = \pm \frac{2}{\sqrt{2}} y$

$\therefore g(x,y) = 0 \Rightarrow x^2 + 2y^2 - 1 = 0$

$\Rightarrow x^2 + 2y^2 = 1$

$\Rightarrow \left[\pm \frac{2}{\sqrt{2}} y\right]^2 + 2y^2 = 1$

$\Rightarrow \frac{4y^2}{2} + 2y^2 = 1$

$\Rightarrow 2y^2 + 2y^2 = 1$

$\Rightarrow 4y^2 = 1 \Rightarrow y^2 = \frac{1}{4} \Rightarrow y = \pm \frac{1}{2}$

$\therefore x = \pm \frac{2}{\sqrt{2}} y \Rightarrow \pm \frac{2}{\sqrt{2}} \times \pm \frac{1}{2} = \pm \frac{1}{\sqrt{2}}$

$\therefore$ The extreme values occur at

(x,y) $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right), \left(\frac{-1}{\sqrt{2}}, \frac{-1}{2}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{2}\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{2}\right)$

Hence the extreme values are

at $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$, $xy = \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{1}{2\sqrt{2}}$

at $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{2}\right)$, $xy = -\frac{1}{\sqrt{2}} \times \frac{-1}{2} = \frac{1}{2\sqrt{2}}$

at $\left(\frac{1}{\sqrt{2}}, \frac{-1}{2}\right)$, $xy = \frac{1}{\sqrt{2}} \times -\frac{1}{2} = -\frac{1}{2\sqrt{2}}$

at $\left(\frac{-1}{\sqrt{2}}, \frac{1}{2}\right)$, $xy = \frac{-1}{\sqrt{2}} \times \frac{1}{2} = \frac{-1}{2\sqrt{2}}$

$\therefore$ extreme values are $\frac{1}{2\sqrt{2}}$ and $-\frac{1}{2\sqrt{2}}$

Find the shortest distance from the origin to the hyperbola $x^2 + 8xy + 7y^2 = 225$?

Let O be the origin and $P(x, y)$ be any point on the hyperbola. Then we have to find the minimum value of $|OP| = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2+y^2}$

$\therefore f(x, y) = x^2 + y^2$

$g(x, y) = x^2 + 8xy + 7y^2 - 225 = 0$.

$\nabla f = 2x\hat{i} + 2y\hat{j}$

$\nabla g = (2x + 8y)\hat{i} + (8x + 14y)\hat{j}$

$\therefore \nabla f = \lambda \nabla g \Rightarrow 2x\hat{i} + 2y\hat{j} = \lambda\left[(2x+8y)\hat{i} + (8x+14y)\hat{j}\right]$

on comparing both sides we get

$2x = \lambda[2x+8y]$ and $2y = \lambda[8x+14y]$

$x = \lambda[x+4y]$ and $y = \lambda[4x+7y]$

$x = \lambda x + 4\lambda y$ and $y = 4x\lambda + 7y\lambda$

$x - \lambda x = 4\lambda y$ and $y - 7y\lambda = 4x\lambda$

$x[1-\lambda] = 4\lambda y$ and $y[1-7\lambda] = 4x\lambda$

$$x = \frac{4\lambda y}{1-\lambda} \quad - (1) \qquad \text{and } y = \frac{4\lambda x}{1-7\lambda} \quad - (2)$$

$$\therefore (1) \Rightarrow x = \frac{4\lambda}{1-\lambda} \times \frac{4\lambda x}{1-7\lambda}$$

$$\Rightarrow x = \frac{16\lambda^2 x}{(1-\lambda)(1-7\lambda)}$$

$$\Rightarrow x - \frac{16\lambda^2 x}{(1-\lambda)(1-7\lambda)} = 0$$

$ab=0$ $a=0$ or $b=0$

$$\Rightarrow x\left[1 - \frac{16\lambda^2}{(1-\lambda)(1-7\lambda)}\right] = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad 1 - \frac{16\lambda^2}{(1-\lambda)(1-7\lambda)} = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad 1 = \frac{16\lambda^2}{(1-\lambda)(1-7\lambda)}$$

$$\Rightarrow x = 0 \quad \text{or} \quad (1-\lambda)(1-7\lambda) = 16\lambda^2$$

$$\Rightarrow x = 0 \quad \text{or} \quad 9\lambda^2 + 8\lambda - 1 = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad \lambda = \frac{-8 \pm \sqrt{(8)^2 - 4\times 9\times -1}}{2\times 9} = \frac{-8 \pm \sqrt{64+36}}{18}$$

$$x = 0 \quad \text{or} \quad \lambda = \frac{-8 \pm 10}{18} = \frac{-8+10}{18}, \frac{-8-10}{18}$$

$x=0$ or $\lambda = \frac{1}{9}$, $\lambda = -1$.

f $x=0$, then ② $\Rightarrow y = \frac{4\lambda \times 0}{1-7\lambda} = 0$.

$\therefore g(x,y) = g(0,0) = 0^2 + 8\times 0\times 0 + 7\times 0 - 225 = -225$.

$\because g(x,y) \neq 0$. so $(0,0)$ is not a Point of the given hyperbola. Hence $x \neq 0$.

f $\lambda = -1$. Then ① $\Rightarrow x = \frac{4\times -1\times y}{1-^-1}$

$\Rightarrow x = -\frac{4y}{2} = -2y$

$\Rightarrow x = -2y$

$\therefore g(x,y) = 0 \Rightarrow x^2 + 8xy + 7y^2 - 225 = 0$

$\Rightarrow (-2y)^2 + 8\times -2y \times y + 7y^2 - 225 = 0$

$\Rightarrow 4y^2 - 16y^2 + 7y^2 - 225 = 0$

$\Rightarrow -5y^2 - 225 = 0$

$\Rightarrow -5y^2 = 225$

$\Rightarrow y^2 = \frac{225}{-5} = -45$

$\Rightarrow y^2 = -45$

No real solution excist. Hence $\lambda \neq -1$.

If $\lambda = 1/9$, ① $\Rightarrow x = \frac{4\times 1/9 \times y}{1-1/9} = \frac{\frac{4y}{9}}{8/9} = \frac{4y}{8} = y/2$

$x = \frac{y}{2}$

$\therefore g(x,y)=0 \Rightarrow \left(\frac{y}{2}\right)^2 + 8 \cancel{xy}\, x\frac{y}{2} + 7y^2 - 225 = 0$

$\Rightarrow \frac{y^2}{4} + 4y^2 + 7y^2 - 225 = 0$

$\Rightarrow \frac{y^2}{4} + 11y^2 - 225 = 0$

$\Rightarrow \frac{y^2 + 44y^2}{4} - 225 = 0$

$\Rightarrow \frac{y^2 + 44y^2}{4} = 225$

$\Rightarrow y^2 + 44y^2 = 900$

$\Rightarrow 45y^2 = 900$

$\Rightarrow y^2 = 20$

$\Rightarrow y = \sqrt{20}$

$\therefore x = \frac{y}{2} \Rightarrow x = \frac{\sqrt{20}}{2}$

$\therefore$ The extreme values occur at $\left(\frac{\sqrt{20}}{2}, \sqrt{20}\right)$

and the shortest distance is $\sqrt{x^2+y^2} \Rightarrow$

$\sqrt{\left(\frac{\sqrt{20}}{2}\right)^2 + (\sqrt{20})^2} = \sqrt{\frac{20}{4} + 20} = \sqrt{25} = 5$

Lagrange Multipliers with Two Constraints

Suppose that $f(x,y,z)$ and $g(x,y,z)$ are differentiable.

To find the local maximum and minimum values of f

subject to the constraints $g_1(x,y,z)=0$ and $g_2(x,y,z)=0$,

find the values of x, y, z and λ that simultaneously

satisfy the equations

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 , \quad g_1(x,y,z)=0, \quad g_2(x,y,z)=0.$$

> Find the maximum and minimum values of the function

$f(x,y,z)=xy+z^2$ on the circle in which the plane $y-x=0$

intersect the sphere $x^2+y^2+z^2=4$.?

$$f(x,y,z)=xy+z^2$$

$$g_1(x,y,z)=0 \Rightarrow y-x=0$$

$$g_2(x,y,z)=0 \Rightarrow x^2+y^2+z^2-4=0.$$

$$\nabla f = y\hat{i}+x\hat{j}+2z\hat{k}$$

$$\nabla g_1 = -\hat{i}+\hat{j}$$

$$\nabla g_2 = 2x\hat{i}+2y\hat{j}+2z\hat{k}.$$

$$\therefore \nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow$$

$$y\hat{i}+x\hat{j}+2z\hat{k} = \lambda[-\hat{i}+\hat{j}] + \mu[2x\hat{i}+2y\hat{j}+2z\hat{k}]$$

$$y\hat{i}+x\hat{j}+2z\hat{k} = -\lambda\hat{i}+\lambda\hat{j}+2x\mu\hat{i}+2y\mu\hat{j}+2z\mu\hat{k}$$

· on comparing both sides we get

$y = -\lambda + 2\mu x$ — ①

$x = \lambda + 2\mu y$ — ②

$2z = 2z\mu$ — ③

① $\Rightarrow \lambda = 2\mu x - y$ — ④

② $\Rightarrow \lambda = x - 2\mu y$ — ⑤

From ④ and ⑤ $2\mu x - y = x - 2\mu y$

$2\mu x + 2\mu y = x + y$

$2\mu[x+y] = x+y$

$2\mu(x+y) - (x+y) = 0$

$(x+y)[2\mu - 1] = 0$

$x+y=0$ or $2\mu - 1 = 0$

$x+y=0$ or $\cancel{2}\,\mu = 1/2$

③ $\Rightarrow 2z - 2z\mu = 0$

$\Rightarrow 2z[1-\mu] = 0$

$\Rightarrow z[1-\mu] = 0$

$\Rightarrow z = 0$ or $1 - \mu = 0$

$\Rightarrow z = 0$ or $\mu = 1$

If $x+y=0$, $g_1(x,y,z)=0$ and $g_2(x,y,z)=0 \Rightarrow$

$\Rightarrow y-x=0$ and $x^2+y^2+z^2-4=0$

$\Rightarrow y=x$ and $x^2+y^2+z^2=4$ — (6)

$\because x=y$, $x+y=0 \Rightarrow x+x=0 \Rightarrow 2x=0 \Rightarrow x=0$.

$\because x=y$, $x=0=y$.

$\therefore$ (6) $\Rightarrow x=y=0$ and $x^2+y^2+z^2=4$

$\Rightarrow x=y=0$ and $0^2+0^2+z^2=4$ $[\because x=y=0]$

$\Rightarrow x=y=0$ and $z^2=4$

$\Rightarrow x=y=0$ and $z=\pm 2$

$\therefore$ Extreme values occur at $(0,0,2)$ and $(0,0,-2)$.

If $z=0$, $g_1(x,y,z)=0$ and $g_2(x,y,z)=0 \Rightarrow$

$\Rightarrow y-x=0$ and $x^2+y^2+0^2-4=0$

$\Rightarrow y-x=0$ and $x^2+y^2=4$

$\Rightarrow y=x$ and $x^2+x^2=4$ $[\because x=y]$

$\Rightarrow y=x$ and $2x^2=4$

$\Rightarrow y=x$ and $x=\pm\sqrt{2}$

$\Rightarrow y=x=\pm\sqrt{2}$,

$\therefore$ Extreme values occur at $(\sqrt{2}, \sqrt{2}, 0)$ and $(-\sqrt{2}, -\sqrt{2}, 0)$,

and the value is

at $(0,0,2)$, $\quad f = 0\times 0 + 2^2 = \underline{\underline{4}}$

at $(0,0,-2)$, $\quad f = 0\times 0 + (-2)^2 = 4$

at $(\sqrt{2}, \sqrt{2}, 0)$, $\quad F = \sqrt{2}\times\sqrt{2} + 0^2 = 2$

at $(-\sqrt{2}, -\sqrt{2}, 0)$, $\quad f = -\sqrt{2}\times -\sqrt{2} + 0^2 = 2$

$\therefore$ max. value is 4 and min. value is 2.