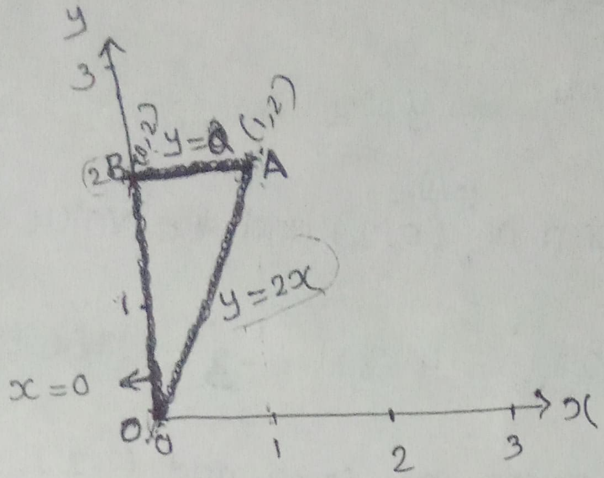
1) Absolute Extrema on Closed Bounded Regions

Let $f(x,y)$ be a continuos function defined on a closed bounded region R. Then f has an 'absolute maximum' value on R at (a,b) in R if $\boxed{f(x,y) \leq f(a,b)}$

and an 'absolute minimum' value on R at (a,b) in R

if $\boxed{f(x,y) \geq f(a,b)}$.

1) Find the absolute maximum and minimum values of $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate in the I^{st} quadrant bounded by the lines $x=0$, $y=2$, $y=2x$?



$$f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$$

$$f_x = 4x - 4$$

$$f_y = 2y - 4$$

$$f_x = 0 \Rightarrow 4x - 4 = 0 \Rightarrow \underline{x = 1}$$

$$f_y = 0 \Rightarrow 2y - 4 = 0 \Rightarrow 2y = 4 \Rightarrow \underline{y = 2}$$

$\therefore f$ attain extreme value at $(1,2)$.

The value of the function at $(1,2) = f(1,2)$

$$= 2\times 1^2 - 4\times 1 + 2^2 - 4\times 2 + 1$$

$$= 2 - 4 + 4 - 8 + 1 = \underline{-5}$$

consider the segment OB.

on OB, $x = 0$.

$$\therefore f(x,y) = f(0,y) = 2\times 0^2 - 4\times 0 + y^2 - 4y + 1$$

$$f(0,y) = y^2 - 4y + 1$$

$$f'(0,y) = 2y - 4$$

$(x,y)=(0,2)$

$f'(0,y)=0 \Rightarrow 2y-4=0$

$\Rightarrow 2y=4 \Rightarrow y=2$

$\therefore$ local extrema occur at $(0,y)=(0,2)$ and the value is

$f(0,2)=\cancel{0}+(2)^2-4\times2+1=4-8+1=-3$.

on OB, the endpoint points are $(0,0)$ and $(0,2)$.

$\therefore f(0,0)=(0)^2-4\times0+1=1$

$f(0,2)=-3$

consider OA. on OA $y=2x$.

$\therefore f(x,y)=f(x,2x)=2x^2-4x+(2x)^2-4\times2x+1$

$=2x^2-4x+4x^2-8x+1$

$f(x,2x)=6x^2-12x+1$

$f'(x,2x)=12x-12$

$f'(x,2x)=0 \Rightarrow 12x-12=0$

$\Rightarrow x=1$

$\therefore$ local extrema occur at $(x,2x)=(1,2)$ and the value is $f(1,2)=2\times(1)^2-4\times1+(2)^2-4\times2+1$

$=2-4+4-8+1$

$=-5$

on OA, the end points are $(0,0)$ and $(1,2)$.

$\therefore f(0,0) = 2\times 0^2 - 4\times 0 + 0^2 - 4\times 0 + 1 = \underline{\underline{1}}$

conside AB. on AB $y=2$.

$\therefore f(x,y) = f(x,2) = 2\times x^2 - 4\times x + (2)^2 - 4\times 2 + 1$

$f(x,2) = 2x^2 - 4x + 4 - 8 + 1$

$f(x,2) = 2x^2 - 4x - 3$

$f'(x,2) = 4x - 4$

$f'(x,2) = 0 \Rightarrow 4x - 4 = 0$

$\Rightarrow x = 1.$

$\therefore$ local extrema occur at $(x,2) = (1,2)$ and the value is

$f(1,2) = -5.$

on AB, the endpoints are $(0,2)$ and $(1,2)$

$\therefore f(0,2) = -3,\ f(1,2) = -5.$

Hence f assume the absolute extreme values are $(0,0), (0,2),$ $(1,2)$. At these points the values of the function are

$f(0,0) = 1,\ f(1,2) = -5,\ f(0,2) = -3.$

From the above list, f attain an absolute maximum at $(0,0)$ and maximum value is 1 and f attain an absolute minimum at $(1,2)$ and minimum value is -5.

Find Positive numbers x, y, z such that $x+y+z=18$ and xyz is maximum ?

Given $x+y+z=18$

$\Rightarrow z = 18-(x+y) = 18-x-y$

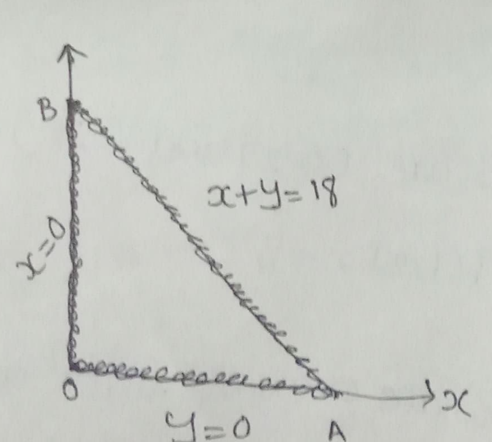
Then $xyz = xy[18-x-y] = 18xy - x^2y - xy^2$.

Let $f(x,y) = 18xy - x^2y - xy^2$.

$\because$ x, y, and z are Positive numbers, $x>0$, $y>0$ and $z>0$.

$z>0 \Rightarrow 18-x-y>0 \Rightarrow 18>x+y \Rightarrow x+y<18$.

Then we have to maximize $f(x,y)$ over $x>0$, $y>0$, $x+y<18$.



$f(x,y) = 18xy - x^2y - xy^2$

$f_x = 18y - 2xy - y^2$

$f_y = 18x - x^2 - 2xy$

$f_x = 0 \Rightarrow 18y - 2xy - y^2 = 0$ — (1)

$f_y = 0 \Rightarrow 18x - x^2 - 2xy = 0 \quad -(2)$

$(1) - (2) \Rightarrow 18y - 18x - y^2 + x^2 = 0$

$\Rightarrow 18y - 18x - (y^2 - x^2) = 0$

$\Rightarrow 18(y - x) - (y^2 - x^2) = 0$

$\Rightarrow 18(y - x) - [(y + x)(y - x)] = 0 \qquad [\because a^2 - b^2 = (a+b)(a-b)]$

$\Rightarrow (y - x)[18 - (y + x)] = 0$

$\Rightarrow (y - x)[18 - y - x] = 0$

$\Rightarrow y - x = 0 \quad \text{or} \quad 18 - y - x = 0.$

$\because z > 0, z = 18 - y - x > 0$. so $18 - y - x \neq 0$

$\therefore y - x = 0 \Rightarrow \underline{\underline{y = x}}$

$\therefore (1) \Rightarrow 18x - 2x^2 - x^2 = 0 \Rightarrow 18x - 3x^2 = 0$

$\Rightarrow 18x = 3x^2$

$\Rightarrow 18 = 3x$

$\Rightarrow x = \frac{18}{3} = 6 //$

$\because x = y, \underline{\underline{y = 6}}$

$z = 18 - x - y = 18 - 6 - 6 = 18 - 12 = 6 //$

Hence xyz attain an extreme value when $x = y = z = 6$.

$f_{xx} = -2y, \quad f_{yy} = -2x, \quad f_{xy} = 18 - 2x - 2y.$

When $x = 6$, $f_{xx} = -2 \times 6 = -12 < 0$.

$f_{xx} f_{yy} - f_{xy}^2 = -24x - 2x - [18 - 2x - 2y]^2$

When $x = y = 6$, $f_{xx} f_{yy} - f_{xy}^2 = -2 \times 6 \times -2 \times 6 - [18 - 12 - 12]^2$

$= -12 \times -12 - [-6]^2$

$= 144 - 36$

$= 108 > 0$

$\because f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$, ~~...~~ $f(x,y)$ and xyz is maximum when $x = y = z = 6$.

) Find the Point $P(x,y,z)$ closest to the origin on the Plane $2x + y - z - 5 = 0$?

$\Rightarrow$ Let O be origin and $P(x,y,z)$ be any Point on the Plane $2x + y - z - 5 = 0$. Then we have to find (x,y,z) so that the function $|\vec{OP}| = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$

$= \sqrt{x^2 + y^2 + z^2}$

is minimum.

Let $f = x^2 + y^2 + z^2$.

$2x + y - z - 5 = 0 \Rightarrow z = 2x + y - 5$

$\therefore f = x^2 + y^2 + [2x + y - 5]^2$

$f_x = 2x + 2(2x+y-5)\times 2 = 2x + 4(2x+y-5)$

$f_x = 2x + 8x + 4y - 20 = 10x + 4y - 20$. (10)

$f_y = 2y + 2(2x+y-5) = 2y + 4x + 2y - 10$

$f_y = 4x + 4y - 10$

$f_x = 0 \Rightarrow 10x + 4y - 20 = 0$

$\Rightarrow 10x + 4y = 20 \quad — (1)$

$f_y = 0 \Rightarrow 4x + 4y - 10 = 0$

$\Rightarrow 4x + 4y = 10 \quad — (2)$

$(1) - (2) \Rightarrow \quad 6x = 10$

$x = \frac{10}{6} = \frac{5}{3}$

$\therefore (1) \Rightarrow 10 \times \frac{5}{3} + 4y = 20$

$\Rightarrow \frac{50}{3} + 4y = 20$

$4y = 20 - \frac{50}{3} = \frac{60-50}{3} = \frac{10}{3}$

$y = \frac{\frac{10}{3}}{4} = \frac{10}{3} \times \frac{1}{4} = \frac{10}{12} = \frac{5}{6}$

$\therefore$ Function attain extreme values at $(5/3, 5/6)$.

$f_{xx} = 10 > 0$. $\quad f_{xy} = 4$.

$f_{yy} = 4$

$f_{xx} f_{yy} - f_{xy}^2 = 10 \times 4 - (4)^2$

$= 40 - 16 = 34 > 0$

$\because f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$, f attain ~~[illegible]~~ minimum value at $(5/3, 5/6)$.

$2x + y - z - 5 = 0 \Rightarrow 2 \times \frac{5}{3} + \frac{5}{6} - z - 5 = 0$

$\Rightarrow \frac{10}{3} + \frac{5}{6} - 5 = z$

$\Rightarrow \frac{20}{6} + \frac{5}{6} - \frac{30}{6} = z$

$= \frac{-5}{6} = z$

$\therefore$ The Points closest to the origin is $(5/3, 5/6, -5/6)$