Non homogeneous linear differential Equation

General form of non homogeneous linear D.E

is $y'' + p(x)y' + Q(x)y = R(x)$.

is a non homogeneous d.E with const

coefficient is the form

$$y'' + ay' + by = R(x) \longrightarrow (1)$$

To find the soln of (1) first find the solution y_c of

the homogeneous equa obtained by setting $R(x)=0$

in eq (1).

The fn y_c is called the complementary fn. Then

find the particular integral y_p and general solut

is given by $y = y_c + y_p$. To find the particular

solution we use the method of undeterminant

coefficient. If $R(x)$ is of the form ke^{px}

choose y_p as ce^{px} where c is the coeffici

to be determin if $R(x)$ is of the form

,

kx^n $(n = 0, 1 \dots)$ choose

y_p as $k_n x^n + k_{n-1} x^{n-1} + \dots + k_1 x + k_0$

If $R(x)$ is $k \cos qx$ or $k \sin qx$ choose

$k_1 \cos qx + k_2 \sin qx$ as y_p.

Subst y_p in the given equa and find the coeffi.

Then. $y = y_c + y_p \rightarrow$ general solution

If $R(x)$ is the sum of the fn described above choose y_p as the sum of correspond particular fn.

1. Solve $y'' - y' - 2y = 6e^x \rightarrow (1)$

homog equa corres to (1) is

$y'' - y' - 2y = 0$.

Charac eq is $m^2 - m - 2 = 0$.

The root is $m = 2$ & $m = -1$

$\therefore$ The complement fn $y_c = c_1 e^{2x} + c_2 e^{-x}$.

$R(x) = 6e^x$.

Let $y_p = ce^x$

sub this in eq (i) we get

$$ce^x - ce^x - 2ce^x = 6e^x$$

$$-2ce^x = 6e^x$$

$$c = -3$$

hence $y_p = ce^x \Rightarrow -3e^x$.

∴ the general eq solut is given by

$$y = y_c + y_p$$

$$y = c_1e^{2x} + c_2e^{-x} - 3e^x.$$

2 $(D_1^2 - 1)y = 2x^2$.

$$y'' - y = 2x^2 \rightarrow (1).$$

here homoge equ is:

$$y'' - y = 0.$$

Chara equ is:

$m^2 - 1 = 0$ ∴ $m = \pm 1$

$$y_c = c_1e^x + c_2e^{-x}.$$

$R(x) = 2x^2$

$y_p = K_2x^2 + K_1x + K_0.$

$y'_p = 2K_2 + K_1$

$y_p'' = 2k_2$

Sub this in eq (1) we get

$$2k_2 - (k_2 x^2 + k_1 x + k_0) = 2x^2$$

equat coeff of x^2 & x.

$K_2 = 2$; $K_1 = 0$.

$2K_2 - K_0 = 0$.

$\therefore K_2 = -2$.

$-4 - K_0 = 0$

$K_0 = -4$.

$y_p = -2x^2 - 4$.

$\therefore$ the general solut is

$$y = c_1 e^{x} + c_2 e^{-x} - 2x^2 - 4$$

3. $y'' - 3y' + 2y = 4x + e^{3x} \rightarrow (1)$.

hom eq is $y'' - 3y' + 2y = 0$.

chara eq is $m^2 - 3m + 2 = 0$.

$m_1 = 2$; $m_2 = 1$

$y_c = C_1 e^{2x} + C_2 e^{x}$.

Now $R(x) = 4x + e^{3x}$.

$y_p = k_1 x + k_0 + c e^{3x}$

$y'_p = k_1 + 3c e^{3x}$

$y''_p = 9c e^{3x}$

sub in eq (1) we get:

$9ce^{3x} - 3(k_1 + 3ce^{3x}) + 2(k_1 x + k_0 + ce^{3x})$
$= 4x + e^{3x}$.

$\Rightarrow -3k_1 + 2k_1 x + 2k_0 + 2ce^{3x} = 4x + e^{3x}$

$2k_1 = 4$ $\qquad$ $k_0 = 0$.
$k_1 = 2$

$2ce^{3x} = e^{3x}$
$2c = 1$ $\therefore$ $c = 1/2$.

$y_p = 2x + \frac{1}{2} e^{3x}$

$\therefore$ the gener solution is

$y = C_1 e^{2x} + C_2 e^{x} + 2x + \frac{1}{2} e^{3x}$

1. $y'' - 3y' + 2y = x^2 + e^{4x}$. $\rightarrow (1)$

hom equ is $y'' - 3y' + 2y = 0$.

chara equ is $m^2 - 3m + 2 = 0$

$m = 1, 2$

$y_c = C_1 e^x + C_2 e^{2x}$.

$R(x) = x^2 + e^{4x}$.

$y_p = k_2 x^2 + k_1 x + k_0 x + c e^{4x}$

$y'_p = 2k_2 x + k_1 + c e^{4x} \cdot 4$.

$y''_p = 2k_2 + 16 c e^{4x}$

Put this in eq (1)

$(2k_2 + 16c e^{4x}) - 3[2k_2 x + k_1 + c e^{4x} \cdot 4)$

$+ 2(k_2 x^2 + k_1 x + k_0 x + c e^{4x}) = x^2 + e^{4x}$

$\Rightarrow 12 c e^{4x} = 1$

$\Rightarrow c = 1/12$

$2k_2 = 1$ $\quad$ $k_1 = 3/2$

$k_2 = 1/2$ $\quad$ $k_0 = 3/4$

$y = c_1 e^{x} + c_2 e^{2x} + \frac{1}{2} x^2 + \frac{3}{2} x + \frac{7}{4} + \frac{1}{12} e^{4x}$

$= c_1 e^{x} + c_2 e^{2x} + \frac{1}{4}\left[2x^2 + 6x + 7\right] + \frac{1}{12} e^{4x}$.

2 $y'' + 4y' + 5y = e^{2x} + \cos 4x \rightarrow (1)$.

homog equa is

$y'' + 4y' + 5y = 0$.

Char equ is $m^2 + 4m + 5 = 0$

$\frac{-4 \pm \sqrt{16 - 20}}{2}$

$= \frac{-4 \pm \sqrt{-4}}{2}$

$= -2 \pm i$

$\Rightarrow -2 \pm i$

here $\alpha = -2$; $\beta = 1$

$y_c = e^{-2x}\left[A_1 \cos x + A_2 \sin x\right]$

$R(x) = e^{2x} + \cos 4x$.

$y_p = Ce^{2x} + k_1 \cos 4x + k_2 \sin 4x$,

$y_p' = Ce^{2x} \times 2 + k_1 - \sin 4x \times 4 + k_2 \cos 4x \times 4$

$y_p'' = 4Ce^{2x} + \cancel{16k\sin} - 16 k_1 \cos 4x$
$+ 16 k_2 \sin 4x$

eq (1) becomes.

$$\left[4Ce^{2x} - 16k_1 \cos 4x - 16k_2 \sin 4x\right] +$$

$$4\left[2Ce^{2x} - 4k_1 \sin 4x + 4k_2 \cos 4x\right]$$

$$+5\left[Ce^{2x} + k_1 \cos 4x + k_2 \sin 4x\right] = e^{2x} + \cos 4x$$

$$\Rightarrow 17Ce^{2x} - 11k_1 \cos 4x - 11k_2 \sin 4x - 16k_1 \sin 4x + 16k_2 \cos 4x = e^{2x} + \cos 4x.$$

$17C = 1$

$C = 1/17$

$(-11k_1 - 16k_2) = 1.$

$(-16k_2 - 16k_1) = 0.$

$-11k_1 - 16k_1 = 0$

$-27k_1 = 1 \Rightarrow k_1 = -1/27$

$+16k_2 - 11k_1 = 1$

$-16k_1 - 11k_2 = 0.$

$176k_2 - 126k_1 = 1$

$-176k_2 - 176k_1 = 0$

$-176k_1 - 176k_1 = 1$

$k_1 = \frac{1}{352}$

$$y = e^{-2x}\left[A_1 \cos x + A_2 \sin x\right] + \frac{1}{17}e^{2x}$$

$$+ \frac{1}{352}(16 \sin 4x - 11 \cos 4x)$$

3. $(D^2+1)y = 10e^x \sin x$

hom eq is $y'' + y = 10e^x \sin x$

Char eq $m^2 + 1 = 10e^x \sin x$.

$m^2 = \pm 1 \therefore m = -i$

$R(x) = 10e^x \sin x$

$R'(x) = 10e^x \cos x + 10e^x \sin x$

$R''(x) = -10e^x \sin x + 10e^x \cos x$

$+ 10e^x \cos x + 10e^x \sin x$.

$= 20e^x \cos x$.

variable part of $R''(x) = e^x \cos x$.

$y_p = k_1 e^x \sin x + k_2 e^x \cos x$,

Che eq $= m^2 + 1 = 0$.

$\Rightarrow m^2 = \pm 1$

$m = -i$

$y_c = A_1 \cos x + A_2 \sin x$.

$y_p' = k_1 e^x \sin x + k_1 e^x \cos x$

$+ k_2 e^x \cos x - k_2 e^x \sin x$

$$y_p'' = k_1 e^x \sin x + k_1 e^x \cos x + k_1 e^x \cos x - k_1 e^x \sin x$$

$$- k_2 e^x \sin x - k_2 e^x \cos x + k_2 e^x \cos x - k_2 e^x \sin x$$

$$= 2k_1 e^x \cos x - 2k_2 e^x \sin x$$

sub this in eq (1)

$$\left[2k_1 e^x \cos x - 2k_2 e^x \sin x\right] + k_1 e^x \sin x + k_2 e^x \cos x$$

$$= 10 e^x \sin x$$

$$-2k_2 + k_1 = 10.$$

$$2 \times \quad 2k_1 + k_2 = 0.$$

$$\begin{aligned} -2k_2 + k_1 &= 10 \quad + \\ 4k_1 + 2k_2 &= 0 \\ \hline 5k_1 &= 10 \\ k_1 &= 2 \end{aligned}$$

$$-2k_2 + 2 = 10$$

$$-2k_2 = 8$$

$$k_2 = -4$$

$$y_p = 2e^x \sin x + -4e^x \cos x$$

$$y = A_1 \cos x + A_2 \sin x + 2e^x \sin x - 4e^x \cos x$$

Find the general solution of $y'' - 2y' + y = e^t + t$

homo eq : $y'' - 2y' + y = 0$.

Char eq : $m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0$

$m = 1, 1$ double root

$y_c = (C_1 + C_2 t)e^t$

$r(x) = e^t + t$

$y_p = ce^t + k_1 t + k_0$

modify with t

$y_p = cte^t + kt + k_0$

modification rule

$y_p = ct^2e^t + kt + k_0$.

$y'_p = c(t^2 + 2t)e^t + k_1$

$y''_p = c(t^2 + 4t + 2)e^t$

$\therefore y'' - 2y' + y = 0$

$\Rightarrow c(t^2+4t+2)e^t - 2(c(t^2+2t)e^t + k_1)$

$+ [ct^2e^t + kt + k_0] = e^t + t$

$= 2ce^t + k_1 t + (k_0 - 2k_1) = e^t + t$

$2C_1 = 1 \quad \therefore C_1 = 1/2$.

$k_1 = 1$; $k_0 - 2k_1 = 0$

$k_0 = 2$

$$y_p = \left(\frac{1}{2}\right) t^2 e^t + t + 2$$

$\therefore$ general solution is

$$y(t) = y_c(t) + y_p(t)$$

$$= (c_1 t + c_2) e^t + \left(\frac{1}{2}\right) t^2 e^t + t + 2$$

1. $y'' + 4y = 8x + 4 \Rightarrow (1)$

homoge eq : $y'' + 4y = 0$

Charac eq : $m^2 + 4 = 0 \Rightarrow m^2 = -4$

$\therefore m = -2i, +2i$

$y(c) = e^{\alpha x}(A \cos \beta x + B \sin \beta x)$

$(\alpha + i\beta = 0 \pm 2i)$

$y(c) = A \cos 2x + B \sin 2x$

$r(x) = 8x + 4$

$y(p) = k_1 x + k_0$

$y'(p) = k_1$

$y''p = 0$

$\therefore$ eq (1) $\Rightarrow 4(k_1 x + k_0) = 8x + 4$

$\Rightarrow 4k_1 x + 4k_0 = 8x + 4$

$4k_1 = 8$; $4k_0 = 4$

$k_1 = 2$; $k_0 = 1$.

$y_p = 2x + 1$

$\therefore$ the solution is $y = y_c$ $\underline{\underline{A \cos 2x + A_2 \sin 2x + 2x + 1}}$

Method of Variation of Parameters

The method of variation of parameters gives a particular solution of y_p of the given equation in the form.

$$y_p(t) = -y_1(t)\int \frac{y_2(t)\, g(t)}{W(y_1, y_2)(t)}\, dt + y_2(t)\int \frac{y_1(t)\, g(t)}{W(y_1, y_2)\, t}\, dt$$

and the general solution of the non homogen equat is

$$y(t) = y_c(t) + y_p(t)$$

Remark

Before applying the above form y'' should be the first term and its coefficient is 1

1. Find the general solution of d.E

$$y'' + y = \csc t$$

homo diff $= y'' + y = 0$

Char equ $= m^2 + 1 = 0 \Rightarrow m^2 = -1$

$m = \pm i$

$$y_c = e^{\alpha x}\left[A_1 \cos\beta x + A_2 \sin\beta x\right]$$

$$= \left[A_1 \cos t + A_2 \sin t\right]$$

$$y_p = -y_1(t)\int \frac{y_2(t)\,g(t)}{W(y_1,y_2)(t)}\,dt + y_2(t)\int \frac{y_1(t)\,g(t)}{W(y_1,y_2)(t)}\,dt$$

$$W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}$$

$$= \underline{1}.$$

$$\therefore y_p = -\cos t\int \frac{\sin t \cdot \csc t}{1}\,dt + \sin t\int \frac{\cos t\,\csc t}{1}\,dt$$

$$= -\cos t\int 1\,dt + \sin t\int \frac{\cos t}{\sin t}\,dt.$$

$$\Rightarrow -t\cos t + \sin t \ln|\sin t|$$

General solution

$y(t) = y_c(t) + y_p(t)$

$= A_1\cos t + A_2\sin t - t\cos t + \sin t \ln|\sin t|$

2 $y'' - 2y' + y = \frac{e^t}{t^3}$

homog eq $= y'' - 2y + y$.

Chara eq $= m^2 - 2m + 1 = 0$.

$(m-1)^2 = 0 \Rightarrow m = \pm 1$

$y_c = C_1 e^t + C_2 t e^t$

$y_1(t) = e^t$; $y_2(t) = te^t$

$$W(y_1, y_2) = \begin{vmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{vmatrix}$$

$= e^t(t+1)e^t - e^t \cdot te^t = \underline{\underline{e^{2t}}}$

$$y_p = -y_1(t)\int \frac{y_2(t)\, g(t)}{W(y_1, y_2)\, t}\, dt + y_2(t)\int \frac{y_1(t)\, g(t)}{W(y_1, y_2)\, t}\, dt$$

$$= -e^t \int \frac{te^t \cdot t^{-3}e^t}{e^{2t}} dt + te^t \int \frac{e^t \cdot t^{-3}e^t}{e^{2t}} dt$$

$$\Rightarrow -e^t \int t^{-2} dt + te^t \int t^{-3} dt$$

$$\Rightarrow -e^t \frac{t^{-1}}{-1} + te^t \frac{t^{-2}}{-2}$$

$$\Rightarrow \frac{e^t}{t} - \frac{e^t}{2t} = \frac{e^t}{2t} //.$$

Hence

$$y(t) = C_1e^t + C_2te^t + \frac{e^t}{2t}$$

3. $t^2y'' - 4ty' + 6 = 21t^{-4}$

$$= y'' - \frac{4}{t}y' + \frac{6}{t^2} = \frac{21}{t^6}$$

$$= y'' - 4t^{-1}y' + 6t^{-2} = 21t^{-6}$$

$$t^2y'' - 4ty' - 6 = 21t^{-4} \longrightarrow (1).$$

which is a Euler equation.

$\therefore$ replace $x = \ln t$.

$$y' = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{1}{t} \frac{dy}{dx}$$

$$y'' = \frac{d^2y}{dt^2} = \frac{d}{dt}\left[\frac{1}{t}\frac{dy}{dx}\right] = \frac{-1}{t^2}\frac{dy}{dx} + \frac{1}{t}\frac{d}{dt}\left(\frac{dy}{dx}\right)$$

$$= \frac{1}{t^2}\frac{d^2y}{dx^2} - \frac{1}{t^2}\frac{dy}{dx}$$

hence eq (1) become.

$$t^3\left[\frac{1}{t^2}\frac{d^2y}{dx^2} - \frac{1}{t^2}\frac{dy}{dx}\right] - 4t\left[\frac{1}{t}\frac{dy}{dx}\right] + 6y = 0.$$

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0.$$

homog eq is $y'' - 5y' + 6 = 0.$

charc eq is $m^2 - 5m + 6 = 0$,

$$m = 2 \text{ or } 3.$$

$$y_c = c_1 e^{2x} + c_2 e^{3x}$$

$$= c_1 t^2 + c_2 t^3$$

$y_1(t) = t^2$; $y_2(t) = t^3$

$$w(y_1, y_2) = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} = 3t^2 \cdot t^2 - 2t \cdot t^3$$

$$\Rightarrow 3t^4 - 2t^4$$

$$= t^4$$

$$y_p = -y_1(t)\int \frac{y_2(t)g(t)}{W(y_1,y_2)}dt + y_2(t)\int \frac{y_1(t)g(t)}{W(y_1,y_2)}dt$$

$$= -t^2\int \frac{t^3 . 21t^{-6}}{t^4}dt + t^3 \int \frac{t^2 - 21t^{-6}}{t^4}dt$$

$$\Rightarrow -21t^2\int t^{-7}dt + 21t^3\int t^{-8}dt$$

$$\Rightarrow -21t^2\frac{t^{-6}}{-6} + 21t^3\frac{t^{-7}}{-7} = \frac{1}{2t^4}$$

$$y(t) = y_c + y_p$$

$$= c_1t^2 + c_2t^3 + \frac{1}{2t^4}$$

n. $y'' + 3y' + y = 3e^x$

hom eq $= y'' + 3y' + y = 0$

char eq $= m^2 + 3m + 1 = 0.$

$$m = \frac{-3 \pm \sqrt{5}}{2}.$$

$$y_c = c_1e^{mx} + c_2e^{mx}$$

$$= c_1e^{\frac{-3\pm\sqrt{5}}{2}x} + c_2e^{\frac{-3+\sqrt{5}}{2}x}.$$

$R(x) = 3e^{x}$

$y_p = ce^{x}$; $y'_p = ce^{x}$; $y''_p = ce^{x}$

Now eq (1)

$ce^{x} + 3ce^{x} + ce^{x} = 3e^{x}$

$5ce^{x} = 3e^{x} \Rightarrow c = 3/5$

$y_p = \frac{3}{5}e^{x}$

$y = y_c + y_p$

$= c_1 e^{\frac{-3+\sqrt{5}}{2}x} + c_2 e^{\frac{-3-\sqrt{5}}{2}x} + \frac{3}{5}e^{x}$.

2 $y'' + 2y' + y = \cos x \rightarrow (1)$

hom eq $= y'' + 2y' + y = 0$

char eq $= m^2 + 2m + 1 = 0$.

$m = -1$

$y_c = (c_1 + c_2 x) e^{mx}$.

$y_c = (c_1 + c_2 x) e^{-x}$

$R(x) = \cos x$

$y_p = k_1 \cos x + k_2 \sin x$

$y'_p = -k_1 \sin x + k_2 \cos x$

$y''_p = -k_1 \cos x + -k_2 \sin x$

Now eq (1)

$$-k_1 \cos x - k_2 \sin x - 2k_1 \sin x + 2k_2 \cos x$$

$$+ k_1 \cos x + k_2 \sin x = \cos x$$

$$2k_2 = 1 \qquad k_1 = 0.$$

$$k_2 = 1/2$$

Now $y_p = k_1 \cos x + k_2 \sin x$

$$= \frac{1}{2} \sin x$$

$$y = y_c + y_p$$

$$= (c_1 + c_2 x)\, e^{-x} + \frac{1}{2} \sin x$$