# Taylor's formula

## Taylor's formula for $f(x,y)$ at the point $(a,b)$

Suppose $f(x,y)$ and its partial derivatives through order $n+1$ are continuous throughout an open rectangular region $R$ centered at a point $(a,b)$. Then throughout $R$.

$$f(x,y) = f(a,b) + \left[h f_x(a,b) + k f_y(a,b)\right] +$$

$$\frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk\, f_{xy}(a,b) + k^2 f_{yy}(a,b)\right] +$$

$$\frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2k\, f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right]$$

$$+ \cdots + \frac{1}{(n+1)!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^{n+1} f.$$

Taylor formula at origin

$$f(x,y) = f(0,0) + x f_x(0,0) + y f_y(0,0) +$$

$$\frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy\, f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

$$+ \frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2y\, f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0)\right.$$

$$\left. + y^3 f_{yyy}(0,0)\right]$$

$$+ \cdots + \frac{1}{(n-1)!}\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right]^{n+1} f$$

1   1 1
2   1 2 1
3   1 3 3 1
4   1 4 6 4 1
5   1 5 10 10 5 1

1. Use Taylor's formula to find a quadratic approxi of $f(x,y) = xe^y$ at the origin. Estimate the error in the approximin if $|x| \le 0.1$ & $|y| \le 0.1$?

$f(x,y) = xe^y$.

quadratic approximation is

$$f(x,y) = f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy\, f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] \rightarrow (1)$$

$f(0,0) = 0 \times e^0 = 0 \times 1 = 0$

$f_x = e^y$ ; $f_x(0,0) = e^0 = 1$.

$f_y = xe^y$ ; $f_y(0,0) = 0 \times e^0 = 0$

$f_{xx} = 0$ ; $f_{xx}(0,0) = 0$.

$f_{yy} = xe^y$ ; $f_{yy}(0,0) = 0 \times e^0 = 0$.

$f_{xy} = e^y$ ; $f(x,y)(0,0) = e^0 = 1$

$(1) \Rightarrow f(x,y) = 0 + x \times 1 + y \times 0.$

$$+ \frac{1}{2!}\left[x^2 \times 0 + 2xy \times 1 + y^2 \times 0\right]$$

$$xe^y = x + \frac{1}{2!}[2xy]$$

$$xe^y = x + \frac{1}{2}[2xy]$$

$$xe^y = x + xy$$

Error is

$$|E(x,y)| = \frac{1}{3!}\left[x^3 f_{xxx} + 3x^2y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right] \rightarrow (2)$$

$f_{xxx} = 0$; $f_{xxy} = 0$; $f_{xyy} = e^y$; $f_{yyy} = xe^y$.

Given $|x| \leq 0.1$; $|y| \leq 0.1$

$$f_{xyy} = e^y \Rightarrow e^y = e^{0.1} = 1.1051$$

$$f_{yyy} = xe^y = 0.1 \times e^{0.1} = 0.1105$$

here max value is $1.105$.

$$|E(x,y)| = \frac{1}{3!}\left[|x^3 f_{xxx} + 3x^2y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}|\right]$$

$$\leq \frac{1}{3!}\left[|x^3||f_{xxx}| + 3|x^2||y||f_{xxy}| + 3|x||y^2||f_{xyy}| + |y^3||f_{yyy}|\right]$$

$$\leq \frac{1}{6}\left[|x|^3|f_{xxx}| + 3|x^2||y|\,|f_{xxy}| + 3|x||y|^2 |f_{xyy}| + |y|^3|f_{yyy}|\right]$$

$$\leq \frac{1}{6}\left[(0.1)^3 \times 1.105 + 3\times(0.1)^2 + 0.1 \times 1.105 + 3\times 0.1 \times (0.1)^2 \times 1.105 + (0.1)^3 \times 1.105\right].$$

$$\leq \frac{1}{6}\left[(0.001\times 1.105) + 3\times 0.01 \times 0.1 \times 1.105 + 3\times 0.1 \times 0.01 \times 1.105 + 0.001 \times 1.105\right]$$

$$\leq \frac{1}{6}\left[0.001105 + 0.003315 + 0.003315 + 0.001105\right]$$

$$\leq \frac{1}{6} \times 0.00884 = \underline{\underline{0.00147}}$$

Find quadratic and cubic approximation of

i) $f(x,y) = y \sin x$

quadratic approximation is

$$f(x,y) = f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

$f(x,y) = y \sin x$.

$f(0,0) = 0 \sin 0 = 0$

$f_x = y \cos x \Rightarrow f_x(0,0) \Rightarrow 0 \times \cos 0 \Rightarrow 0$.

$f_y = \sin x \Rightarrow f_y(0,0) \Rightarrow \sin 0 = 0$

$f_{xx} = -y \sin x \Rightarrow f_{xx}(0,0) = 0$.

$f_{yy} = \cancel{\cos x} \sin x \Rightarrow f_{yy}(0,0) = 0$

$f_{xy} = \frac{\partial}{\partial x} \sin x = \cos x \; ; \; f_{xy}(0,0) = 1$.

$(1) \Rightarrow 0 + x \times 0 + y \times 0 + \frac{1}{2!}\left[x^2 \cdot 0 + 2xy \times 1 + 0\right]$.

$\Rightarrow \frac{1}{2} 2xy. = xy.$

cubic approximation is:

$f(x,y) = f(0,0) + x f_x(0,0) + y f_y(0,0)$.

$+ \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$

$+ \frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right] \rightarrow (2)$

$f_{xxx} = -y\cos x$ ; $f(0,0) = 0$

$f_{yyy} = \sin x$ ; $f(0,0) = 0$

$f_{xxy} = -\sin x$ ; $f(0,0) = 0$

$f_{yyx} = \cos x$ ; $f(0,0) = $ "

$f_{xyy} = 0$ ; $f(0,0) = 0$

(2) $\Rightarrow$

$$0+0+0+\frac{1}{2}(2xy)+\frac{1}{3!}\left[x^3\times 0+3x^2y\times 0 + 3xy^2\times 0 + y^3\times 0\right].$$

$\Rightarrow xy$

ii) $f(x,y) = e^x \ln(1+y)$.

$f(0,0) = e^0 \ln(1+0) \Rightarrow .0$

$f_x = e^x \ln(1+y)$, $f(0,0) = 0$.

$f_{xx} = e^x \ln(1+y)$, $f(0,0) = 0$.

$f_{xxx} = e^x \ln(1+y)$, $f(0,0) = 0$.

$f_y = \dfrac{e^x}{1+y}$, $f(0,0) = 1$

$f_{yy} = \dfrac{-e^x}{(1+y)^2}$, $f(0,0) = -1$

$f_{yyy} = \dfrac{2e^x}{(1+y)^3}$, $f(0,0) = 2$

$f_{xy} = \dfrac{e^x}{1+y}$, $f(0,0) = 1$.

$f_{xxy} = \dfrac{e^x}{y+1}$, $f(0,0) = 1$

$f_{xyy} = \dfrac{-e^x}{(1+y)^2}$, $f(0,0) = -1$.

Cubic

approximation

$f(x,y) = f(0,0) + x f_x(0,0) + y f_y(0,0)$

$+ \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$

$+ \frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right]$

$\Rightarrow 0 + x \times 0 + y \times 1 + \frac{1}{2!}\left[x^2 \times 0 + 2xy \times 1 + y^2 \times -1\right]$

$+ \frac{1}{3!}\left[x^3 \times 0 + 3x^2y \times 1 + 3xy^2 \times -1 + y^3 \times 2\right]$

$\Rightarrow y + \frac{1}{2!}\left[2xy - y^2\right] + \frac{1}{3!}\left[3x^2y - 3xy^2 + 2y^3\right]$

$\Rightarrow y + \frac{1}{2!}\left[2xy - y^2\right] + \frac{1}{6}\left[3x^2y - 3xy^2 + 2y^3\right]$

$\therefore$ the quadratic approximation is.

$y + \frac{1}{2!}\left[2xy - y^2\right]$

2 Find quadratic approximation of $f(x,y) = \cos x \cos y$ of origin. Estimate error if $|x| \le 0.1$ & $|y| \le 0.1$

$f(x,y) = \cos x \cos y$

$f(0,0) = \cos 0 \cos 0 = 1$

$f_x = -\sin x \cos y \quad f_x(0,0) = 0$

$f_y = -\sin y \cos x \quad f(0,0) = 0$

$f_{xx} = -\cos x \cos y, \quad f(0,0) = -1$

$f_{yy} = -\cos y \cos x, \quad f(0,0) = -1$

$f_{xy} = \frac{\partial}{\partial x} -\sin y \cos x = +\sin y \sin x = 0$

$f(x,y) = 1 + x \times 0 + y \times 0 + \frac{1}{2!}\left[x^2 \times -1 + 2xy \times 0 + y^2 \times -1\right]$

$= 1 + \frac{1}{2!}\left[-x^2 + 0 - y^2\right] \Rightarrow 1 - \frac{1}{2!}\left[x^2 + 0 + y^2\right]$

$\Rightarrow 1 - \frac{1}{2!}(x^2 + y^2)$

Error is

$$|E(x,y)| = \frac{1}{3!}\left[|x^3||f_{xxx}| + 3|x^2y||f_{xxy}| + 3|xy^2||f_{xyy}| + |y^3||f_{yyy}|\right]$$

$|f_{xxx}| = |\sin x \ \cos y| . \cancel{f(0,0)} \le |x| = 1$

$|f_{yyy}| = \cos x \ \sin y \le 1$

$|f_{xxy}| = \cos x \ \sin y \le 1.$

$|f_{xyy}| = +\cos y \ \sin x \le 1$

$|x| \le 0.1$ & $|y| \le 0.1$

$$|E(x,y)| \le \frac{1}{3!}\left[(0.1)^3 \times 1 + 3 \times (0.1)^2 \times (0.1) \times 1 + 3 \times 0.1 \times (0.1)^2 \times 1 + (0.1)^3 \times 1\right].$$

$$\le \frac{1}{6}\left[0.001 + 3 \times 0.001 + 3 \times 0.001 + 0.001\right]$$

$$\le 0.00133$$